Formal equivalences	Season	2
	Episode	04
	Time frame	1 period

Prerequisites :
- Truth tables.
- Logical operators $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
- Notion of formal equivalence.

Objectives :
- Build some complex truth tables.
- Compare truth tables and find formal equivalences.

Materials :
- *Fact sheet with a few logical transformation rules.*
- *Nine pairs of formally equivalent propositions involving three propositions p, q, r.*
- *Complete truth tables of each proposition.*
- *Beamer with a few logical transformation rules.*

1 – Build a truth table — 15 mins

Students work in pairs. Each pair is given a logical proposition and has to build its truth table. The teacher checks each truh table before proceeding to the second part.

2 – Find formal equivalences — 10 mins

Students mingle (still working by pairs) to find a proposition formally equivalent to the one they have.

3 – Formal derivation — 20 mins

The teacher gives a few logical transformation rules, with a few examples. Then students, working in teams of four, have to use them to transform one of their two propositions into the other.

4 – Challenge — 10 mins

Students have to reduce the following property so that there are brackets and no conditional. To prove that their result is good, they have to build the truth tables of the two propositions.

$$\neg\left((p \wedge \neg r) \vee (\neg p \vee r)\right) \longrightarrow \neg\left(p \wedge (\neg q \vee r)\right)$$

De Morgan's laws

Theorem 1

$$\neg(p \wedge q) \iff (\neg p \vee \neg q) \text{ and } \neg(p \vee q) \iff (\neg p \wedge \neg q)$$

Proof. Let's look at the truth table of each proposition.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
0	0	0	1	1	1	1
0	1	0	1	1	0	1
1	0	0	1	0	1	1
1	1	1	0	0	0	0

As $\neg(p \wedge q)$ and $\neg p \vee \neg q$ have the same truth table, they are formally equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

As $\neg(p \vee q)$ and $\neg p \wedge \neg q$ have the same truth table, they are formally equivalent. □

Conditional and biconditional

Theorem 2

$$(p \to q) \iff (\neg q \to \neg p) \iff (\neg p \vee q)$$

Proof. The first formal equivalence has been proven in the previous session, so we just have to look at the truth tables of $p \to q$ and $\neg p \vee q$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

As $p \to q$ and $\neg p \vee q$ have the same truth table, they are formally equivalent. □

Theorem 3

$$(p \leftrightarrow q) \iff (\neg q \leftrightarrow \neg p)$$

Proof. Once again, we just have to look at the truth tables of the two propositions.

p	q	$p \leftrightarrow q$	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$
0	0	1	1	1	1
0	1	0	1	0	0
1	0	0	0	1	0
1	1	1	0	0	1

As $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ have the same truth table, they are formally equivalent. □

Document 1 Pairs of formally equivalent propositions

$$(p \wedge q) \to r$$

$$\neg r \to (\neg p \vee \neg q)$$

$$(\neg p \wedge \neg r) \to q$$

$$\neg q \to (p \vee r)$$

$$\neg p \to \neg(q \vee r)$$

$$\neg(\neg q \wedge \neg r) \to p$$

$$(\neg p \to q) \to \neg r$$

$$r \to (\neg p \land \neg q)$$

$$(p \lor q) \to (p \land r)$$

$$(\neg p \lor \neg r) \to (\neg q \land \neg p)$$

$$\neg p \leftrightarrow (q \land r)$$

$$(\neg q \lor \neg r) \leftrightarrow p$$

$$(p \to q) \lor (p \to \neg r)$$

$$p \to (q \lor \neg r)$$

$$(p \land q) \leftrightarrow (\neg p \lor r)$$

$$(p \land \neg r) \leftrightarrow (\neg p \lor \neg q)$$

$$(\neg p \lor p) \to (q \land r)$$

$$(\neg q \lor \neg r) \to (p \land \neg p)$$

Document 2 Pairs of formally equivalent propositions with complete truth tables

$(p \wedge q) \to r$

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

$\neg r \to (\neg p \vee \neg q)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg p \vee \neg q$	$\neg r \to (\neg p \vee \neg q)$
0	0	0	1	1	1	1	1
0	0	1	1	1	0	1	1
0	1	0	1	0	1	1	1
0	1	1	1	0	0	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	0	1	1
1	1	0	0	0	1	0	0
1	1	1	0	0	0	0	1

$(\neg p \wedge \neg r) \to q$

p	q	r	$\neg p$	$\neg r$	$\neg p \wedge \neg r$	$(\neg p \wedge \neg r) \to q$
0	0	0	1	1	1	0
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	0	0	1
1	0	0	0	1	0	1
1	0	1	0	0	0	1
1	1	0	0	1	0	1
1	1	1	0	0	0	1

$\neg q \to (p \vee r)$

p	q	r	$\neg q$	$p \vee r$	$\neg q \to (p \vee r)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	1	1 $\neg p \to \neg(q \vee r)$	1
1	0	1	1	1	1
1	1	0	0	1	1
1	1	1	0	1	1

$\neg p \to \neg(q \vee r)$

p	q	r	$\neg p$	$q \vee r$	$\neg(q \vee r)$	$\neg p \to \neg(q \vee r)$
0	0	0	1	0	1	1
0	0	1	1	1	0	0
0	1	0	1	1	0	0
0	1	1	1	1	0	0
1	0	0	0	0	1	1
1	0	1	0	1	0	1
1	1	0	0	1	0	1
1	1	1	0	1	0	1

$\neg(\neg q \wedge \neg r) \to p$

p	q	r	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$\neg(\neg q \wedge \neg r)$	$\neg(\neg q \wedge \neg r) \to p$
0	0	0	1	1	1	0	1
0	0	1	1	0	0	1	0
0	1	0	0	1	0	1	0
0	1	1	0	0	0	1	0
1	0	0	1	1	1	0	1
1	0	1	1	0	0	1	1
1	1	0	0	1	0	1	1
1	1	1	0	0	0	1	1

$(\neg p \to q) \to \neg r$

p	q	r	$\neg p$	$\neg r$	$\neg p \to q$	$(\neg p \to q) \to \neg r$
0	0	0	1	1	0	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	0	1	0
1	0	0	0	1	1	1
1	0	1	0	0	1	0
1	1	0	0	1	1	1
1	1	1	0	0	1	0

$r \to (\neg p \wedge \neg q)$

p	q	r	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$r \to (\neg p \wedge \neg q)$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	0	0	0
1	0	0	0	1	0	1
1	0	1	0	1	0	0
1	1	0	0	0	0	1
1	1	1	0	0	0	0

$(p \vee q) \rightarrow (p \wedge r)$

p	q	r	$p \vee q$	$p \wedge r$	$(p \vee q) \rightarrow (p \wedge r)$
0	0	0	0	0	1
0	0	1	0	0	1
0	1	0	1	0	0
0	1	1	1	0	0
1	0	0	1	0	0
1	0	1	1	1	1
1	1	0	1	0	0
1	1	1	1	1	1

$(\neg p \vee \neg r) \rightarrow (\neg q \wedge \neg p)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg p \vee \neg r$	$\neg q \wedge \neg p$	$(\neg p \vee \neg r) \rightarrow (\neg q \wedge \neg p)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	0	1	1	1
0	1	0	1	0	1	1	0	0
0	1	1	1	0	0	1	0	0
1	0	0	0	1	1	1	0	0
1	0	1	0	1	0	0	0	1
1	1	0	0	0	1	1	0	0
1	1	1	0	0	0	0	0	1

$\neg p \leftrightarrow (q \wedge r)$

p	q	r	$\neg p$	$q \wedge r$	$\neg p \leftrightarrow (q \wedge r)$
0	0	0	1	0	0
0	0	1	1	0	0
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	0	1
1	0	1	0	0	1
1	1	0	0	0	1
1	1	1	0	1	0

$(\neg q \vee \neg r) \leftrightarrow p$

p	q	r	$\neg q$	$\neg r$	$\neg q \vee \neg r$	$(\neg q \vee \neg r) \leftrightarrow p$
0	0	0	1	1	1	0
0	0	1	1	0	1	0
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	1	1	1	1
1	0	1	1	0	1	1
1	1	0	0	1	1	1
1	1	1	0	0	0	0

$(p \to q) \vee (p \to \neg r)$

p	q	r	$\neg r$	$p \to q$	$p \to \neg r$	$(p \to q) \vee (p \to \neg r)$
0	0	0	1	1	1	1
0	0	1	0	1	1	1
0	1	0	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	1	1
1	0	1	0	0	0	0
1	1	0	1	1	1	1
1	1	1	0	1	0	1

$p \to (q \vee \neg r)$

p	q	r	$\neg r$	$q \vee \neg r$	$p \to (q \vee \neg r)$
0	0	0	1	1	1
0	0	1	0	0	1
0	1	0	1	1	1
0	1	1	0	1	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	1	1	1
1	1	1	0	1	1

$(p \wedge q) \leftrightarrow (\neg p \vee r)$

p	q	r	$\neg p$	$p \wedge q$	$\neg p \vee r$	$(p \wedge q) \leftrightarrow (\neg p \vee r)$
0	0	0	1	0	1	0
0	0	1	1	0	1	0
0	1	0	1	0	1	0
0	1	1	1	0	1	0
1	0	0	0	0	0	1
1	0	1	0	0	1	0
1	1	0	0	1	0	0
1	1	1	0	1	1	1

$(p \wedge \neg r) \leftrightarrow (\neg p \vee \neg q)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$p \wedge \neg r$	$\neg p \vee \neg q$	$(p \wedge \neg r) \leftrightarrow (\neg p \vee \neg q)$
0	0	0	1	1	1	0	1	0
0	0	1	1	1	0	0	1	0
0	1	0	1	0	1	0	1	0
0	1	1	1	0	0	0	1	0
1	0	0	0	1	1	1	1	1
1	0	1	0	1	0	0	1	0
1	1	0	0	0	1	1	0	0
1	1	1	0	0	0	0	0	1

$(\neg p \vee p) \rightarrow (q \wedge r)$

p	q	r	$\neg p$	$\neg p \vee p$	$q \wedge r$	$(\neg p \vee p) \rightarrow (q \wedge r)$
0	0	0	1	1	0	0
0	0	1	1	1	0	0
0	1	0	1	1	0	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	0
1	1	0	0	1	0	0
1	1	1	0	1	1	1

$(\neg q \vee \neg r) \rightarrow (p \wedge \neg p)$

p	q	r	$\neg p$	$\neg q$	$\neg r$	$\neg q \vee \neg r$	$p \wedge \neg p$	$(\neg q \vee \neg r) \rightarrow (p \wedge \neg p)$
0	0	0	1	1	1	1	0	0
0	0	1	1	1	0	1	0	0
0	1	0	1	0	1	1	0	0
0	1	1	1	0	0	0	0	1
1	0	0	0	1	1	1	0	0
1	0	1	0	1	0	1	0	0
1	1	0	0	0	1	1	0	0
1	1	1	0	0	0	0	0	1