Arithmetic sequences	Season	02
	Episode	08
	Time frame	4 periods

Prerequisites :

Objectives :
- Discover the concept of arithmetic sequence.
- Find out the main formulae about arithmetic sequences.

Materials :
- *Answer sheet for the team work.*
- *Lesson about arithmetic sequences.*
- *Exercises about arithmetic sequences.*
- *Terms from seven different sequences.*
- *Beamer*

1 – Matching game — 10 mins

Papers with numbers are handed out to the class. Students mingle to find the other numbers that could be part of the same sequence. First terms are specified with a star on the paper.

2 – Team work — 45 mins

Working in the teams from the previous part, students have to fill an answer sheet about their sequence.

3 – Lesson — 30 mins

The main results about arithmetic sequences are shown with a beamer.

4 – Exercises — Remaining time

Exercises about arithmetic sequences have to be done in groups of 3 or 4 students.

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1. Write in the cells below the first five terms of your sequence, in the correct order.

2. Write in the cells below the next five terms of your sequence.

3. Write in the cells below the terms of your sequence whose indices are given. For example, in the first cell you have to write the 20th term in your sequence.

20		25		50		100	

4. What is the common difference d between two consecutive terms in your sequence ?

$d =$

5. Let's note a_1 the first term in your sequence, a_2 the second term and so on. Give the *notation* – not the value – of the term next to each of the terms below.

a_6		a_{12}		a_{153}		a_n	

6. Find a relation between any term a_n and the next term.

7. Find a relation between a term a_n, the common difference d and the first term a_1.

8. Use the formula you found in the previous question to compute directly these terms.

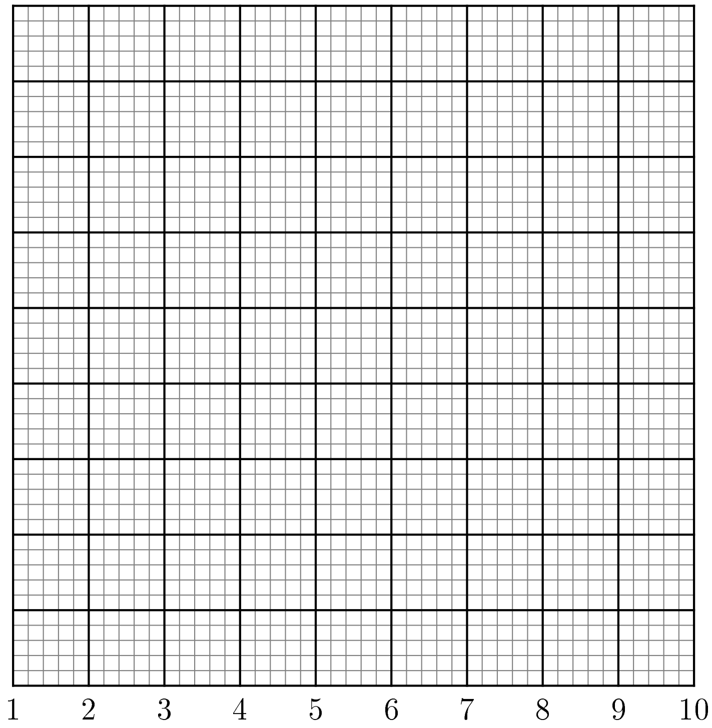
a_{200}		a_{250}		a_{500}		a_{1000}	

9. Find a relation between any term a_n and the term a_2.

10. Find a relation between any term a_n and the term a_5.

11. Find a relation between any two terms a_n and a_m.

12. Place the first ten terms of your sequence on the graph below. To do so, choose a convenient scale on the y-axis.



13. What do you notice about the graph of this sequence ?

14. Compute the sum of the first five terms of your sequence.

15. Find a relation between the sum of the first five terms, the number of terms, the first term and the fifth term.

16. Find a relation between the sum of the first ten terms, the number of terms, the first term and the tenth term.

17. Find a relation between the sum of the first n terms, the number of terms, the first term and the n-th term.

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1 Definition and criterion

A sequence of numbers (a_n) is arithmetic if the difference between two consecutive terms is a constant number. Intuitively, to go from one term to the next one, we always add the same number.

Definition 1 Arithmetic sequence

A sequence of numbers (a_n) is arithmetic if, for any positive integer n, $a_{n+1} - a_n = d$ where d is a fixed real number, called the *common difference* of the sequence. We can also write that

$$a_{n+1} = a_n + d.$$

This equality is called the *recurrence relation* of the sequence.

Proposition 1 Graph of an arithmetic sequence

The graph of an arithmetic sequence is a straight line.

2 Relations between terms

Proposition 2 Explicit definition

For any positive integer n,

$$a_n = a_1 + (n-1) \times d.$$

This equality is called the *explicit definition* of the sequence.

Proof. First, this equality is true when $n = 1$, as $a_1 = a_1 + 0 \times d = a_1 + (1-1) \times d$. Then, suppose that it is true for a value $n = k$, meaning that $a_k = a_1 + (k-1) \times d$. Then, from the definition of the sequence, $a_{k+1} = a_k + d = a_1 + (k-1) \times d + d = a_1 + k \times d$. So the formula is true for $n = k+1$ too. So it's true for $n = 0$, $n = 1$, $n = 2$, $n = 3$, etc, for all values of n.

Proposition 3 Relation between two terms

For any two positive integers n and m,

$$a_n = a_m + (n-m) \times d.$$

Proof. From the explicit definition of the sequence (a_n), $a_n = a_1 + (n-1) \times d$ and $a_m = a_1 + (m-1) \times d$, so $a_n - a_m = (a_1 + (n-1) \times d) - (a_1 + (m-1) \times d) = n \times d - m \times d = (n-m)d$. Therefore, $a_n = a_m + (n-m) \times d$.

3 Limit when n approaches $+\infty$

Theorem 1 Limit of an arithmetic sequence

The limit of an arithmetic sequence (a_n) of common difference d
- is equal to $+\infty$ when $d > 0$;
- is equal to $-\infty$ when $d < 0$;
- is equal to a_1, trivially, if $d = 0$.

Proof.
- Suppose that $a_1 > 0$ and $d > 0$, and consider any real number K. Then, the inequation $a_n > K$, or $a_1 + (n-1)d > K$, is solved by any positive integer n such that $n > \frac{K-a_1}{d} + 1$. This means that for any real number K, there exist some integer N such that for any $n \geqslant N$, $a_N > K$. This is exactly the definition of the fact that $\lim a_n = +\infty$.
- Suppose that $a_1 > 0$ and $d < 0$, and consider any real number K. Then, the inequation $a_n < K$, or $a_1 + (n-1)d < K$, is solved by any positive integer n such that $n > \frac{K-a_1}{d} + 1$. This means that for any real number K, there exist some integer N such that for any $n \geqslant N$, $a_N < K$. This is exactly the definition of the fact that $\lim a_n = -\infty$.
- The last situation, when $d = 0$, is obvious, as all terms are equal to a_1.

4 Sums of consecutive terms

Theorem 2 Sum of consecutive terms

Let (a_n) be an arithmetic sequence, The sum S_n of all the terms between a_1 and a_n, $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$, or more precisely $S = \sum_{i=1}^{n} a_i$, is given by the formula

$$S = n \times \frac{a_1 + a_n}{2}.$$

Proof. The trick to prove this formula is to write the sum in two different ways, one based on the first term, the other based on the last term. Using the formula of proposition 1.2, we have, on one hand

$$\begin{aligned} S_n &= a_1 + a_2 + \ldots + a_{n-1} + a_n \\ S_n &= a_1 + a_1 + d + \ldots + a_1 + (n-2)d + a_1 + (n-1)d \end{aligned}$$

and on the other hand

$$\begin{aligned} S_n &= a_1 + a_2 + \ldots + a_{n-1} + a_n \\ S_n &= a_n - (n-1)d + a_n - (n-2)d + \ldots + a_n - d + a_n. \end{aligned}$$

When we add these two expression of S_n, all terms involving d are cancelled and we end up with as many times the sum $a_1 + a_n$ are there were of terms in S_n, so

$$\begin{aligned} 2S_n &= n(a_m + a_n) \\ S_n &= n \times \frac{a_1 + a_n}{2}. \end{aligned}$$

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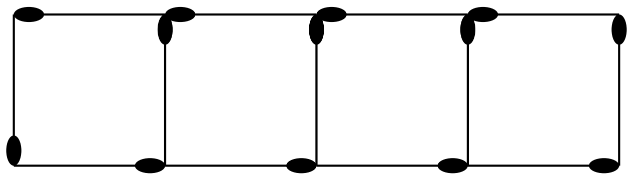
8.1 A single square is made from 4 matchsticks. Two squares in a row needs 7 matchsticks and 3 squares in a row needs 10 matchsticks. This process defines a sequence. Determine, for this sequence,

1. the first term ;
2. the common difference ;
3. the formula for the general term ;
4. how many matchsticks are in a row of 25 squares.



8.2 A triangular number is a natural number such that the shape of an equilateral triangle can be formed by that number of points. For $n \geqslant 1$, let u_n be the difference between the n-th triangular number and the previous one.

1. Find the six first triangular numbers.
2. Give the 5 first terms of the sequence (u_n). What kind of sequence could it be ?
3. Write the six first triangular numbers using consecutive terms of the sequence (u_n).
4. Deduce a formula to find directly the n-th trianguler number.
5. Use this formula to compute the 36th triangular number. Is this number famous for other reasons ?

8.3 The third term of an arithmetic sequence is -7 and the 7^{th} term is 9. Determine :

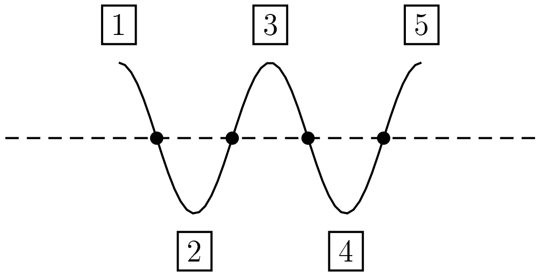
1. the first term a_1 and the common difference d ;
2. the 51^{st} term.

8.4 In an arithmetic sequence, the first and seventh terms are x^2 and $6 + x - 5x^2$, respectively. If the common difference is x, determine the possible values of x.

8.5 The twelfth term of an arithmetic sequence is 5, and the common difference between successive terms is 3. Determine which term has a value of 47.

8.6 In a given arithmetic sequence, the first term is 2, the last term is 29 and the sum of all the terms is 155. Find the common difference.

8.7 A horizontal line intersects a piece of string at four points and divides it into five parts, as shown below.



If the piece of string is intersected in this way by 19 parallel lines, each of which intersects it at four points, find the number of parts into which the string will be divided.

8.8 These two problems appear on the Rhind Papyrus. This papyrus was named after Alexander Henry Rhind, a Scottish antiquarian, who purchased the papyrus in 1858 in Luxor, Egypt ; it was apparently found during illegal excavations in or near the Ramesseum (the memorial temple of Pharaoh Ramesses II). The British Museum, where the papyrus is now kept, acquired it in 1864. It is one of the two well-known Mathematical Papyri along with the Moscow Mathematical Papyrus.

Problem 40. Divide 100 hekats of barley among 5 men so that the common difference is the same and so that the sum of the two smallest is 1/7 the sum of the three largest.

Problem 64. Divide 10 hekats of barley among 10 men so that the common difference is 1/8 of a hekat of barley.

Document 1 Cards for the matching game

* 5	8	11	14
* 7	2	−3	−8
* 17	17,5	18	18,5
* 13	63	113	163
17	−13	19	* 0.3
0.6	0.9	1.2	1.5

* 3.7	3.6	3.5	3.4
3.3	* −5	−7.5	−10
−12.5	−15	* −41	−61
−81	−101		