# Episode 08 – Arithmetic sequences

European section – Season 2

# Definition and criterion

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$$a_{n+1} = a_n + d.$$

This equality is called the *recurrence relation* of the sequence.

# Relations between terms

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So the formula is true for $n = k + 1$ too. So it's true for $n = 0$, $n = 1$, $n = 2$, $n = 3$, etc, for all values of $n$. □

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From the explicit definition of the sequence $(a_n)$,
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$a_n - a_m = (a_1 + (n - 1) \times d) - (a_1 + (m - 1) \times d) = n \times d - m \times d = (n - m)d$.

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Therefore, $a_n = a_m + (n-m) \times d$. □

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- *is equal to $-\infty$ when $d < 0$ ;*
- *is equal to $a_1$, trivially, if $d = 0$.*

# Limit when $n$ approaches $+\infty$

**Proof.**

- Suppose that $a_1 > 0$ and $d > 0$, and consider any real number $K$. Then, the inequation $a_n > K$, or $a_1 + (n-1)d > K$, is solved by any positive integer $n$ such that $n > \frac{K - a_1}{d} + 1$. This means that for any real number $K$, there exist some integer $N$ such that for any $n \geq N$, $a_N > K$. This is exactly the definition of the fact that $\lim a_n = +\infty$.

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- Suppose that $a_1 > 0$ and $d < 0$, and consider any real number $K$. Then, the inequation $a_n < K$, or $a_1 + (n-1)d < K$, is solved by any positive integer $n$ such that $n > \frac{K-a_1}{d} + 1$. This means that for any real number $K$, there exist some integer $N$ such that for any $n \geq N$, $a_N < K$. This is exactly the definition of the fact that $\lim a_n = -\infty$.

**Proof.**

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- Suppose that $a_1 > 0$ and $d < 0$, and consider any real number $K$. Then, the inequation $a_n < K$, or $a_1 + (n-1)d < K$, is solved by any positive integer $n$ such that $n > \frac{K-a_1}{d} + 1$. This means that for any real number $K$, there exist some integer $N$ such that for any $n \geq N$, $a_N < K$. This is exactly the definition of the fact that $\lim a_n = -\infty$.
- The last situation, when $d = 0$, is obvious, as the sequence is constant, with all terms equal to $a_1$.

□

# Sums of consecutive terms

**Theorem**

*Let $(a_n)$ be an arithmetic sequence, The sum $S_n$ of all the terms between $a_1$ and $a_n$, $S_n = a_1 + a_2 + \ldots + a_{n-1} + a_n$, or more precisely $S = \sum_{i=1}^{n} a_i$, is given by the formula*

$$S = n \times \frac{a_1 + a_n}{2}.$$

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\begin{aligned}
S_n &= a_1 + a_2 + \ldots + a_{n-1} + a_n \\
S_n &= a_1 + a_1 + d + \ldots + a_1 + (n-2)d + a_1 + (n-1)d
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and on the other hand

$$\begin{aligned} S_n &= a_1 + a_2 + \ldots + a_{n-1} + a_n \\ S_n &= a_n - (n-1)d + a_n - (n-2)d + \ldots + a_n - d + a_n. \end{aligned}$$

# Sums of consecutive terms

**Proof. (continued).**

When we add these two expression of $S_n$, all terms involving $d$ are cancelled and we end up with as many times the sum $a_1 + a_n$ are there were of terms in $S_n$, so

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$$\begin{aligned} 2S_n &= n(a_m + a_n) \\ S_n &= n \times \frac{a_1 + a_n}{2}. \end{aligned}$$

□