Geometric sequences	Season	02
	Episode	09
	Time frame	3 periods

Prerequisites :

Objectives :
- Discover the concept of geometric sequence.
- Use the main formulae about geometric sequences.

Materials :
- *Lesson about geometric sequences.*
- *Exercises about geometric sequences.*
- *Beamer*

1 – Lesson (30 mins)

The main results about geometric sequences are shown with a beamer.

2 – Exercises (Remaining time)

Exercises about geometric sequences have to be done in groups of 3 or 4 students.

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1 Definition and criterion

A sequence of numbers (b_n) is geometric if the quotient between two consecutive terms is a constant number. Intuitively, to go from one term to the next one, we always multiply by the same number.

Definition 1 Geometric sequence

A sequence of numbers (b_n) is geometric if, for any positive integer n, $\frac{b_{n+1}}{b_n} = q$ where q is a fixed real number, called the *common ratio* of the sequence. We can also write that $b_{n+1} = b_n \times q$. This equality is called the *recurrence relation* of the sequence.

2 Relations between terms

Proposition 1 Explicit definition

For any positive integer n, $b_n = b_1 \times q^{n-1}$. This equality is called the *explicit definition* of the sequence.

Proof. First, this equality is true when $n = 0$, as $b_1 = b_1 \times q^0 = b_1 \times q^{1-1}$. Then, suppose that it is true for a value $n = k$, meaning that $b_k = b_1 \times q^{k-1}$. Then, from the definition of the sequence, $b_{k+1} = b_k \times q = b_1 \times q^{k-1} \times q = b_1 + \times q^k$. So the formula is true for $n = k + 1$ too. So it's true for $n = 0$, $n = 1$, $n = 2$, $n = 3$, etc, for all values of n.

Proposition 2 Relation between two terms

For any two positive integers n and m, $b_n = b_m \times q^{n-m}$.

Proof. From the explicit definition of the sequence (b_n), $b_n = b_1 \times q^{n-1}$ and $b_m = b_1 \times q^{m-1}$, so $\frac{b_n}{b_m} = \frac{b_1 \times q^{n-1}}{b_m = b_1 \times q^{m-1}} = \frac{q^{n-1}}{q^{m-1}} = q^{n-m}$. Therefore, $b_n = b_m \times q^{n-m}$.

3 Limit when n approaches $+\infty$

Theorem 1 Limit of a geometric sequence

The limit of a geometric sequence (b_n) of common ratio q and first term b_1
- is equal to 0, trivially, if $b_1 = 0$;
- is equal to b_1, trivially, if $r = 1$;
- is equal to $+\infty$ when $b_1 > 0$ and $q > 1$;
- is equal to $-\infty$ when $b_1 < 0$ and $q > 1$;
- is equal to 0 if $q \in]-1;1[$;
- doesn't exist when $r \leqslant -1$.

In the last situation, the sequence is said to be *divergent*.

4 Sums of consecutive terms

Theorem 2 Sum of consecutive terms

Let (b_n) be an geometric sequence, The sum S of the n first consecutive terms, defined as $S = b_1 + b_2 + \ldots + b_{n-1} + b_n$, or more precisely $S = \sum_{i=1}^{n} b_i$, is given by the formula

$$S = b_1 \frac{1-q^n}{1-q}.$$

Proof. Using the formula of proposition 2.2, we can write S as

$$\begin{aligned} S &= b_1 + b_1 \times q + \ldots + b_1 \times q^{n-2} + b_1 \times q^{n-1} \\ &= b_1 \times (1 + q + \ldots + q^{n-2} + q^{n-1}). \end{aligned}$$

But, by a simple expansion, we see that $(1 + q + \ldots + q^{n-2} + q^{n-1})(1-q) = 1 - q^n$ and so

$$S = b_1 \frac{1-q^n}{1-q}.$$

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9.1 You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75° F, what will be the temperature of the hot tub after 3 hours, to the nearest tenth of a degree ?

9.2 A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours ?

9.3 A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. Find the amount of radioactive material in the sample at the beginning of the 7th day.

9.4 Here are the first three levels in a school group telephoning tree.

Teacher

Student 1 Student 2

Student 3 Student 4 Student 5 Student 6

At what level are all 53 students in the group contacted ?

9.5 Consider the geometric sequences d and d starting like this : $b = \{\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}, \ldots\}$ and $d = \{\frac{7}{3}, \frac{7}{6}, \frac{7}{12}, \frac{7}{24}, \frac{7}{48}, \ldots\}$.

1. Compute the common ratio q for each sequence.
2. Find recursive definitions for these two geometric sequences.
3. Find the formula of the n-th term for each sequence.
4. Compute the 15-th term of each sequence.

9.6 Bill and Steve decide to buy the same computer. They don't have the money at their disposal yet, so the seller offers two types of credits. In each case, they have to pay a certain amount at the time of the actual sell, then the remainder of the 2000 euros in monthly instalments.
Bill chooses to pay 80 euros first, then monthly instalments of 160 euros for each of the 12 following months. Steve chooses to pay 125 euros first, and the following instalments with a monthly increase of 3 % over the 11 following months. On the twelfth month, he pays whatever is left of the 2000 euros.

Part A – Bill's choice

Let's note u_1 the initial amount paid by Bill and u_n the total amount paid after n month. So, $u_1 = 80$ and u_2 is the total amount he has paid at the end of the first month.

1. Compute u_2 and u_3.
2. **a.** What kind of sequence is (u_n) ? Explain.
   b. Write u_n as a function of n.

Part B – Steve's choice

Let's note v_1 the initial amount paid by Steve and v_n the total amount paid on the n-th month (with n between 2 and 11), rounded to the closest integer. So, $v_1 = 125$ and $v_2 = 129$.

1. Compute the value of v_2 rounded to the closest integer.

2. What kind of sequence is (v_n) ? Explain.

3. What is the total amount paid by Steve at the end of the 11-th month ? What must he pay for the 12-th month ?

4. From what moment are Steve's monthly instalments greater then Bill's ?

9.7 Write the formula for the n-th term of the geometric sequences with the following :

1. $u_5 = 18750$ and $q = 5$;

2. $u_5 = 3$ and $q = \frac{1}{2}$;

3. $u_8 = 472392$ and $q = 3$;

4. $u_5 = 15625$ and $q = -5$;

5. $u_{11} = 20480$ and $u_8 = 5120$;

6. $u_9 = 98415$ and $u_{12} = 2657205$;

7. $u_5 = 54432$ and $u_8 = 11757312$.

9.8 Let (c_n) be the sequence defined by

$$\begin{cases} c_1 & = & \dfrac{1}{3} \\ c_{n+1} & = & \dfrac{n+1}{3n} c_n \end{cases}$$

1. Compute the terms from c_1 to c_6.

2. Let (b_n) be the sequence defined by

$$b_n = \frac{c_n}{n}.$$

Prove that (b_n) is geometric and give its common ratio.

3. Give the explicit formula of (b_n) and deduce the one of (c_n).

4. Compute the sum $\sum_{k=1}^{20} b_k$.

5. Determine the limit of the sequence (b_n).

6. Admitting that for any natural number n, $n < 2^n$, prove that

$$0 \leqslant c_n < \left(\frac{2}{3}\right)^n.$$

7. Deduce from the previous question the limit of the sequence (c_n).