Session 09 – Geometric sequences

European section – Season 2

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Definition Geometric sequence

A sequence of numbers (b_n) is geometric if, for any positive integer n, $\frac{b_{n+1}}{b_n} = q$ where q is a fixed real number, called the *common ratio* of the sequence. We can also write that $b_{n+1} = b_n \times q$. This equality is called the *recurrence relation* of the sequence.

Proposition Explicit definition

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Therefore, $b_n = b_m \times q^{n-m}$.

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- doesn't exist when $r \leq -1$.

In the last situation, the sequence is said to be *divergent*.

Proof.

- If $b_1 = 0$ or if $r = 1$, the result is obvious : in both cases, the sequence is constant !

Proof.

- If $b_1 > 0$ and $q > 1$, consider any real number K. The inequation $b_n > K$, or $b_1 \times q^{n-1} > K$ is equivalent to

$$\begin{aligned} q^{n-1} &> \frac{K}{b_1} \\ \ln(q^{n-1}) &> \ln\left(\frac{K}{b_1}\right) \\ (n-1)\ln q &> \ln K - \ln b_1 \\ (n-1) &> \frac{\ln K - \ln b_1}{\ln q} \\ n &> \frac{\ln K - \ln b_1}{\ln q} + 1 \end{aligned}$$

This means that for any real number K, there exist some integer N such that for any $n \geq N$, $b_N > K$. This is exactly the definition of the fact that $\lim b_n = +\infty$.

Proof.

- If $b_1 < 0$ and $q > 1$, consider any real number K. The inequation $b_n < K$, or $b_1 \times q^{n-1} < K$ is equivalent to

$$\begin{aligned}
q^{n-1} &> \frac{K}{b_1} \\
\ln(q^{n-1}) &> \ln\left(\frac{K}{b_1}\right) \\
(n-1)\ln q &> \ln K - \ln b_1 \\
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This means that for any real number K, there exist some integer N such that for any $n \geq N$, $b_N < K$. This is exactly the definition of the fact that $\lim b_n = -\infty$.

Proof.

- If $q \in]-1;1[$, consider any positive number ε. The inequation $|b_n| < \varepsilon$, or $|b_1| \times |q|^{n-1} < \varepsilon$ is equivalent to

$$\begin{aligned} |q|^{n-1} &< \frac{\varepsilon}{|b_1|} \\ \ln(|q|^{n-1}) &< \ln\left(\frac{\varepsilon}{|b_1|}\right) \\ (n-1)\ln|q| &< \ln\varepsilon - \ln|b_1| \\ (n-1) &> \frac{\ln\varepsilon - \ln|b_1|}{\ln|q|} \\ n &> \frac{\ln\varepsilon - \ln|b_1|}{\ln|q|} + 1 \end{aligned}$$

This means that for any positive real number ε, there exists some integer N such that for any $n \geq N$, $|b_N| < \varepsilon$. This is exactly the definition of the fact that $\lim b_n = 0$.

Proof.

- Finally when $r \leq -1$, the sequence is alternating between positive and negative terms, whose absolute values approach $+\infty$. So the sequence has no limit.

Theorem Sum of consecutive terms

Let (b_n) be an geometric sequence, The sum S of the n first consecutive terms, defined as $S = b_1 + b_2 + \ldots + b_{n-1} + b_n$, or more precisely $S = \sum_{i=1}^{n} b_i$, is given by the formula

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$$S = b_1 \frac{1-q^n}{1-q}.$$

Proof. Using the formula of proposition 2.2, we can write S as

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$$S = b_1 \frac{1 - q^n}{1 - q}.$$