The golden sequence	Season	02
	Episode	11
	Time frame	2 periods

Prerequisites : Geometric sequences, the golden ratio

Objectives :
- Use the properties of geometric sequences on an particularly interesting example.
- Discover some properties of the Golden Ratio.

Materials :
- *Task sheet.*
- *Correction of the task sheet.*
- *Beamer (may not be needed).*

1 – Pair work

2 periods

Working in pairs, students have to answer a series of questions about the golden sequence, the only sequence which is at the same time a Fibonacci sequence and a geometric sequence.

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The Golden number is the irrational real number defined as

$$\varphi = \frac{1+\sqrt{5}}{2}.$$

This famous number is strongly related to the Fibonacci sequence. It is also interesting to study the sequence defined as follows :

$$\begin{cases} b_1 &= \frac{1}{\varphi} \\ b_{n+1} &= b_n \times \varphi \end{cases}$$

Part A – Working with approximate values

1. What kind of sequence is (b_n) ? Give the explicit definition of b_n.
2. Write the first seven terms of the sequence in the simplest way, using φ and without doing any computation.
3. Give an approximate value to 3 DP of the first seven terms of the sequence.
4. Find the first value of n for which $b_n > 100$.
5. Check with the approximate values that for any integer n lower than or equal to 5,

$$b_n + b_{n+1} = b_{n+2}.$$

Part B – Working with exact values.

In this part, no approximate value must be used, so the calculator will be useless. All answers must be given in the form $a + b\sqrt{5}$ where a and b are rational numbers. For example, φ will be written as

$$\varphi = \frac{1}{2} + \frac{1}{2}\sqrt{5}.$$

1. Check that $-\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) = 1$ and deduce the exact value of $b_1 = \frac{1}{\varphi}$.
2. Compute the first seven terms of the sequence (b_n) and give the answers as explained at the beginning of this part.
3. Check, using the exact values, that for any integer n lower than 5, $b_n + b_{n+1} = b_{n+2}$.
4. Deduce the probable values of b_8, b_9, and b_{10}.
5. Check that $\frac{1}{\varphi} + 1 = \varphi$ and deduce that for any positive integer n,

$$\varphi^n + \varphi^{n+1} = \varphi^{n+2}.$$

   What property of the sequence b does it prove ?

Part C – The sum of the first n terms

1. In a table, compare the approximate values (to 3DP) of the sums $b_1 + b_2$, $b_1 + b_2 + b_3$ and $b_1 + b_2 + b_3 + b_4$ with the values of φ^2, φ^3 and φ^4.
2. Deduce from the previous question a possible formula for $b_1 + b_2 + \ldots + b_n$.
3. Prove the formula.

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Part A – Working with approximate values

1. From its definition, we can say that (b_n) is a geometric sequence with first term $\frac{1}{\varphi}$ and common ratio φ. The explicit definition of b_n is therefore

$$b_n = \frac{1}{\varphi} \times \varphi^{n-1} = \varphi^{n-2}.$$

2. Using the explicit formula, we get $b_1 = \frac{1}{\varphi}$, $b_2 = \varphi^0 = 1$, $b_3 = \varphi^1 = \varphi$, $b_4 = \varphi^2$, $b_5 = \varphi^3$, $b_6 = \varphi^4$, $b_7 = \varphi^5$.

3. Using the previous answers and a calculator, we find $b_1 \simeq 0.618$, $b_2 = 1$, $b_3 \simeq 1.618$, $b_4 \simeq 2.618$, $b_5 \simeq 4.236$, $b_6 \simeq 6.854$, $b_7 \simeq 11.090$.

4. The first value of n such that $b_n > 100$ is $n = 12$.

5. For $n = 1$, $b_1 + b_2 = 0.618 + 1 = 1.618 = b_3$.
For $n = 2$, $b_2 + b_3 = 1 + 1.618 = 2.618 = b_4$.
For $n = 3$, $b_3 + b_4 = 1.618 + 2.618 = 4.236 = b_5$.
For $n = 4$, $b_4 + b_5 = 2.618 + 4.236 = 6.854 = b_6$.
For $n = 5$, $b_5 + b_6 = 4.236 + 6.854 = 11.090 = b_7$.

Part B – Working with exact values.

1.

$$\begin{aligned} -\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) &= -\left(\left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2\right) \\ &= -\left(\frac{1}{4} - \frac{5}{4}\right) \\ &= -\left(-\frac{4}{4}\right) \\ &= 1 \end{aligned}$$

This means that $-\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) \times \varphi = 1$, so

$$b_1 = \frac{1}{\varphi} = -\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) = -\frac{1}{2} + \frac{1}{2}\sqrt{5}.$$

2. As we've seen previously, $b_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$. Also, it's obvious that $b_2 = 1 = 1 + 0\sqrt{5}$ and we know that $b_3 = \varphi = \frac{1}{2} + \frac{1}{2}\sqrt{5}$.Let's look at the next terms.
$b_4 = \varphi^2 = \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right)^2 = \frac{1}{4} + \frac{1}{2}\sqrt{5} + \frac{5}{4} = \frac{3}{2} + \frac{1}{2}\sqrt{5}$.
$b_5 = \varphi^3 = \varphi^2 \times \varphi = \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) \times \left(\frac{1}{2} + \frac{1}{2}\sqrt{5}\right) = \frac{3}{4} + \frac{3}{4}\sqrt{5} + \frac{1}{4}\sqrt{5} + \frac{5}{4} = 2 + 1\sqrt{5}$.
$b_6 = \varphi^4 = (\varphi^2)^2 = \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)^2 = \frac{9}{4} + \frac{3}{2}\sqrt{5} + \frac{5}{4} = \frac{7}{2} + \frac{3}{2}\sqrt{5}$.
$b_7 = \varphi^5 = \varphi^3 \times \varphi^2 = \left(2 + 1\sqrt{5}\right) \times \left(\frac{3}{2} + \frac{1}{2}\sqrt{5}\right) = 3 + \sqrt{5} + \frac{3}{2}\sqrt{5} + \frac{5}{2} = \frac{11}{2} + \frac{5}{2}\sqrt{5}$.

3. For $n=1$, $b_1+b_2=-\frac{1}{2}+\frac{1}{2}\sqrt{5}+1+0\sqrt{5}=\frac{1}{2}+\frac{1}{2}\sqrt{5}=b_3$.
For $n=2$, $b_2+b_3=1+0\sqrt{5}+\frac{1}{2}+\frac{1}{2}\sqrt{5}=\frac{3}{2}+\frac{1}{2}\sqrt{5}=b_4$.
For $n=3$, $b_3+b_4=\frac{1}{2}+\frac{1}{2}\sqrt{5}+\frac{3}{2}+\frac{1}{2}\sqrt{5}=2+1\sqrt{5}=b_5$.
For $n=4$, $b_4+b_5=\frac{3}{2}+\frac{1}{2}\sqrt{5}+2+1\sqrt{5}=\frac{7}{2}+\frac{3}{2}\sqrt{5}=b_6$.
For $n=5$, $b_5+b_6=2+1\sqrt{5}+\frac{7}{2}+\frac{3}{2}\sqrt{5}=\frac{11}{2}+\frac{5}{2}\sqrt{5}=b_7$.

4. We can expect that :
$b_8=b_6+b_7=\frac{7}{2}+\frac{3}{2}\sqrt{5}+\frac{11}{2}+\frac{5}{2}\sqrt{5}=9+4\sqrt{5}$
$b_9=b_7+b_8=\frac{11}{2}+\frac{5}{2}\sqrt{5}+9+4\sqrt{5}=\frac{29}{2}+\frac{13}{2}\sqrt{5}$
$b_{10}=b_8+b_9=9+4\sqrt{5}+\frac{29}{2}+\frac{13}{2}\sqrt{5}=\frac{47}{2}+\frac{21}{2}\sqrt{5}$.

5. We proved in the first question that $\frac{1}{\varphi}=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$, so

$$\frac{1}{\varphi}+1=-\frac{1}{2}+\frac{1}{2}\sqrt{5}+1=\frac{1}{2}+\frac{1}{2}\sqrt{5}=\varphi.$$

Multiplying this equation by φ^{n+1}, we deduce that for any positive integer n,

$$\begin{aligned}\varphi^{n+1}\times\left(\frac{1}{\varphi}+1\right) &= \varphi^{n+1}\times\varphi\\ \frac{\varphi^{n+1}}{\varphi}+\varphi^{n+1} &= \varphi^{n+1}\times\varphi\\ \varphi^{n}+\varphi^{n+1} &= \varphi^{n+2}\end{aligned}$$

Using the explicit formula of b_n, this proves that for any positive integer n, $b_{n+2}+b_{n+3}=b_{n+4}$ and so, as its also true for b_1 and b_2,

$$b_n+b_{n+1}=b_{n+2}.$$

Part C – The sum of the first n terms

1. Comparison between the two sequences :

b_1+b_2	1.618	φ^2	2.618
$b_1+b_2+b_3$	3.236	φ^3	4.236
$b_1+b_2+b_3+b_4$	5.854	φ^4	6.854

2. From the results in the previous table, it seems that $b_1+b_2+\ldots+b_n=\varphi^n-1$.

3. Using the formula for the sum of the terms of a geometric sequence, we get

$$b_1+b_2+\ldots+b_n = \frac{1}{\varphi}\times\frac{1-\varphi^n}{1-\varphi}$$

As we know that $\frac{1}{\varphi}+1=\varphi$, we can say that $\frac{1}{\varphi}=\varphi-1$, and so

$$\begin{aligned}b_1+b_2+\ldots+b_n &= (\varphi-1)\times\frac{1-\varphi^n}{1-\varphi}\\ b_1+b_2+\ldots+b_n &= -(1-\varphi)\times\frac{1-\varphi^n}{1-\varphi}\\ b_1+b_2+\ldots+b_n &= -(1-\varphi^n)\\ b_1+b_2+\ldots+b_n &= \varphi^n-1\end{aligned}$$