Session 11 – The golden sequence

European section – Season 2

The Golden sequence

The Golden sequence is the geometric sequence with first term $\frac{1}{\varphi}$ and common ratio φ.
The successive terms of this sequence are

$$b_1 = \frac{1}{\varphi} \simeq 0.618 \quad b_2 = \varphi^0 = 1 \quad b_3 = \varphi \simeq 1.618$$
$$b_4 = \varphi^2 \simeq 2.618 \quad b_5 = \varphi^3 \simeq 4.236 \quad b_6 = \varphi^4 \simeq 6.854$$
$$b_7 = \varphi^5 \simeq 11.090$$

The Golden sequence in the set $\mathbf{Q}[\sqrt{5}]$

The set $\mathbf{Q}[\sqrt{5}]$ is the set of all numbers of the form $a + b\sqrt{5}$, where a and b are fractions.
The terms of the Golden sequence are all in this set. The expressions for the first seven numbers are :

$$b_1 = \frac{1}{\varphi} = -\frac{1}{2} + \frac{1}{2}\sqrt{5} \quad b_2 = 1 = 1 + 0\sqrt{5}$$
$$b_3 = \varphi = \frac{1}{2} + \frac{1}{2}\sqrt{5} \quad b_4 = \varphi^2 = \frac{3}{2} + \frac{1}{2}\sqrt{5}$$
$$b_5 = \varphi^3 = 2 + 1\sqrt{5} \quad b_6 = \varphi^4 = \frac{7}{2} + \frac{3}{2}\sqrt{5}$$
$$b_7 = \varphi^5 = \frac{11}{2} + \frac{5}{2}\sqrt{5}$$

It's interesting to consider the sum of the two coefficients of each term :

$$\begin{aligned}
b_1 &= \tfrac{1}{\varphi} - \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} &&: \quad -\tfrac{1}{2} + \tfrac{1}{2} = 0 \\
b_2 &= 1 = 1 + 0\sqrt{5} &&: \quad 1 + 0 = 1 \\
b_3 &= \varphi = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{5} &&: \quad \tfrac{1}{2} + \tfrac{1}{2} = 1 \\
b_4 &= \varphi^2 = \tfrac{3}{2} + \tfrac{1}{2}\sqrt{5} &&: \quad \tfrac{3}{2} + \tfrac{1}{2} = 2 \\
b_5 &= \varphi^3 = 2 + 1\sqrt{5} &&: \quad 2 + 1 = 3 \\
b_6 &= \varphi^4 = \tfrac{7}{2} + \tfrac{3}{2}\sqrt{5} &&: \quad \tfrac{7}{2} + \tfrac{3}{2} = 5 \\
b_7 &= \varphi^5 = \tfrac{11}{2} + \tfrac{5}{2}\sqrt{5} &&: \quad \tfrac{11}{2} + \tfrac{5}{2} = 8
\end{aligned}$$

These are the terms of the Fibonacci sequence !

The relations between the terms

Using the expressions in $\mathbf{Q}[\sqrt{5}]$, it's easy to see that

$$\varphi = 1 + \frac{1}{\varphi}.$$

Multiplying by φ this equation, we get

$$\varphi^2 = \varphi + 1$$

which proves that the Golden Ratio satisfies the quadratic equation

$$x^2 = x + 1.$$

The relations between the terms

Multiplying the equation $\varphi^2 = \varphi + 1$ by φ^n, we prove that for any integer n

$$\varphi^{n+2} = \varphi^{n+1} + \varphi^n.$$

Therefore, we can conclude that for any integer n, the terms of the sequence satisfy the equality

$$b_{n+2} = b_{n+1} + b_n.$$

SA2, Incidently, this is the same relation as in the Fibonacci sequence. This sequence is the only one that is a geometric sequence and a Fibonacci-like sequence.

The sum of consecutive terms

Using the formula for the sum of consecutive terms in a geometric sequence we get

$$\sum_{i=1}^{n} b_i = b_1 + b_2 + \ldots + b_n$$

$$\sum_{i=1}^{n} b_i = \frac{1}{\varphi} + 1 + \ldots + \varphi^{n-2}$$

$$\sum_{i=1}^{n} b_i = \frac{1}{\varphi} \times \frac{1-\varphi^n}{1-\varphi}$$

$$\sum_{i=1}^{n} b_i = \frac{1}{\varphi} \times \frac{\varphi^n - 1}{\varphi - 1}$$

The sum of consecutive terms

But we know that $\varphi = 1 + \frac{1}{\varphi}$, so $\frac{1}{\varphi} = \varphi - 1$ and

$$\sum_{i=1}^{n} b_i = (\varphi - 1) \times \frac{\varphi^n - 1}{\varphi - 1}$$

$$\sum_{i=1}^{n} b_i = \varphi^n - 1$$

And now for something completely different. . .