Cross sections and coordinates	Season	2
	Episode	17
	Time frame	2 periods

Prerequisites : Cross sections of a cube.

Objectives :
- Work in a three dimensional coordinate.
- Study equations of lines and planes.

Materials :
- *Task sheet.*
- *Cut-out cubes.*
- *Lesson.*
- *Beamer.*

1 – Cartesian equations and planes — 20 mins

Working in pairs students have to answer a series of questions about some equations of planes in 3D.

2 – Parametric equations and lines — 20 mins

Working in pairs students have to answer a series of questions about some parametric equations of lines in 3D.

3 – Lesson — 15 mins

The teacher explains the main concepts of cartesian equations and parametric equations in 3D. The methods to find intersections of planes with lines are also introduced.

4 – Using coordinates to find cross sections — 1 period

Students are still working in pairs. They to draw corss sections of the cube using coordinates. While doing so, they have to master the following techniques :
- find a cartesian equation of a plane knowig three non-collinear points;
- find parametric equations describing a line;
- find the coordinates of the intersection of a plane and a line.

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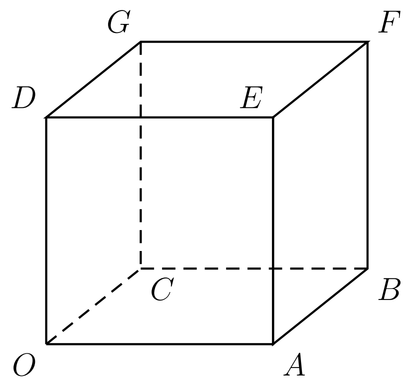
	Document	Task sheet

For this activity we consider a cube $OABCDEFG$ with side $1cm$, as pictured on the right.
Using the point O as the origin and the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ for directions and units, we define a three dimensional cartesain system. The position of any point in the 3D space, and in particular any point on the cube, can be described with a triplet of numbers (x, y, z) : the coordinates of the point. For example, the coordinates of the point O are $(0, 0, 0)$.



Preliminary question : Give the coordinates of all the cube vertices in this system.

Part 1 – Cartesian equations and planes

1. Find all the vertices of the cube such that $z = 0$. Does this equations define a line? If not so, what is the set of the points on the cube such that $z = 0$.

2. Describe in the same way the set of points on the cube such that $x = 0$ and the set of points such that $z = 1$.

3. Give the cartesian equations of the 8 faces of the cube.

4. Find all the vertices of the cube such that $x = y$ then do a quick sketch of the set of the points inside the cube such that $x = y$.

5. Same question for the sets such that

a. $x + y = 1$;
b. $x + y + z = 1$;
c. $x + 2z = 1$;
d. $x + y - z = 1$;
e. $2x + 2y + 2z = 1$;
f. $x + 2y + z = 1$.

6. What seems to be the general form of a cartesian equation of a plane in a 3D coordinate system?

7. **a.** Find a cartesian equation of the plane passing through the points O, F and $N(1, O, \frac{1}{2})$.

b. Find the coordinates of the only point on this plane such that $x = 0$ and $y = 1$.

c. Do a quick sketch of that set in the cube.

8. **a.** Find a cartesian equation of the plane passing through the points A, C and $R(\frac{1}{2}, O, 1)$ and F.

b. Find the coordinates of the only point on this plane such that $x = 0$ and $z = 1$.

c. Do a quick sketch of that set in the cube.

Part 2 – Parametric equations and lines

1. **a.** Describe with a simple sentence the particularity of the coordinates of all the points on the line (OA).

b. Translate the properties you found in the previous question into two simple equalities.

c. One of the coordinates may not be referred to in your previous answers. Why?

2. **a.** Find three points on the cube such that

$$\begin{cases} x &= 1 \\ y &= 1 \\ z &= t \end{cases}$$

where t can be any real number.

b. What is the set of all points on the cube defined by these parametric equations. Draw it.

3. Find a set of parametric equations describing the lines (OC) and (OD), then the lines (DG) and (FG).

4. **a.** Find three points on the cube such that

$$\begin{cases} x &= t \\ y &= t \\ z &= 0 \end{cases}$$

where t can be any real number.

b. What is the set of all points on the cube defined by these parametric equations. Draw it.

5. **a.** Find two vertices of the cube such that

$$\begin{cases} x &= 1-t \\ y &= t \\ z &= t \end{cases}$$

where t can be any real number.

b. Find the coordinates of the point in this set when $t = \frac{1}{2}$, then place it in the cube.

c. What is the set of all points on the cube defined by these parametric equations. Draw it.

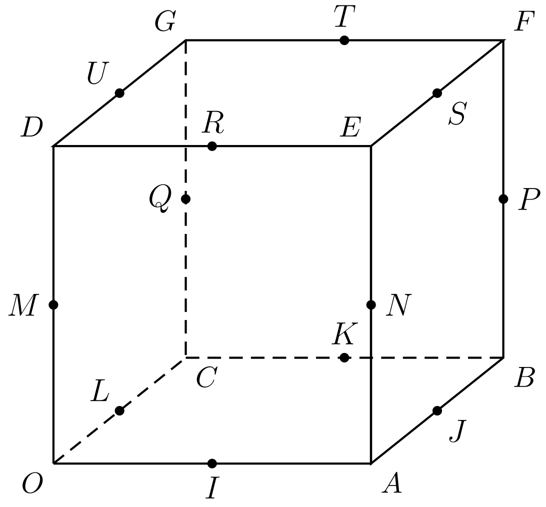
6. Find out the set of all points such that

$$\begin{cases} x &= \frac{1}{2} \\ y &= t \\ z &= 1-t \end{cases}$$

where t can be any real number. Draw it.

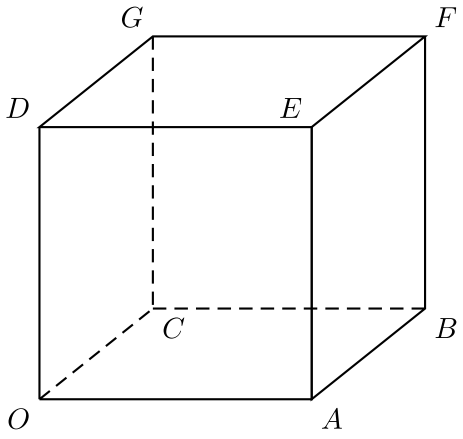
Part 3 – Using coordinates to find cross sections

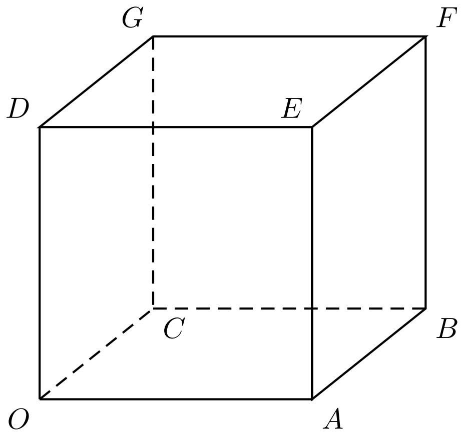
In this part, we name the midpoints of all the edges of the cube as shown in the picture below.

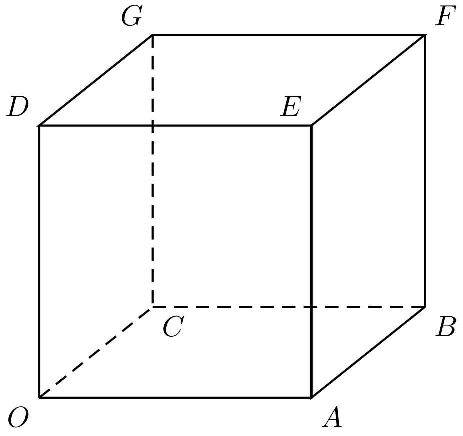


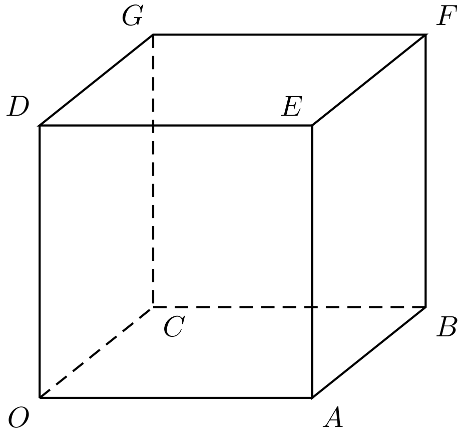
1. Give the parametric equations of all the edges of the cube. You will use these equations often in this part.
2. In this question we wnat to find the section of the cube by the plane (AEL).
   a. Find a cartesian equation of the plane (AEL).
   b. Find the coordinates of the intersection of the plane (AEL) with each edge of the plane, whenever it exists.
   c. Place the points found in the previous question and draw the section of the cube.
3. In this question we wnat to find the section of the cube by the plane (IJU).
   a. Find a cartesian equation of the plane (IJU).
   b. Find the coordinates of the intersection of the plane (IJU) with each edge of the plane, whenever it exists.
   c. Place the points found in the previous question and draw the section of the cube.
4. In this question we wnat to find the section of the cube by the plane (IQS).
   a. Find a cartesian equation of the plane (IQS).
   b. Find the coordinates of the intersection of the plane (IQS) with each edge of the plane, whenever it exists.
   c. Place the points found in the previous question and draw the section of the cube.

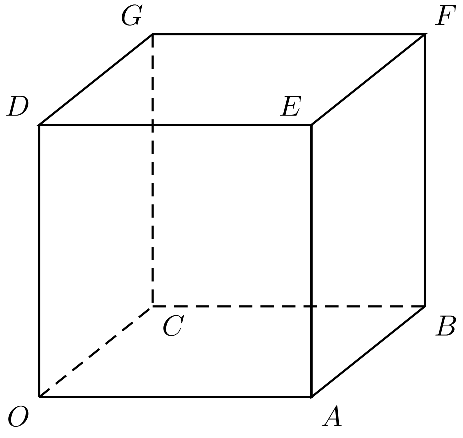
Document 1 Cubes

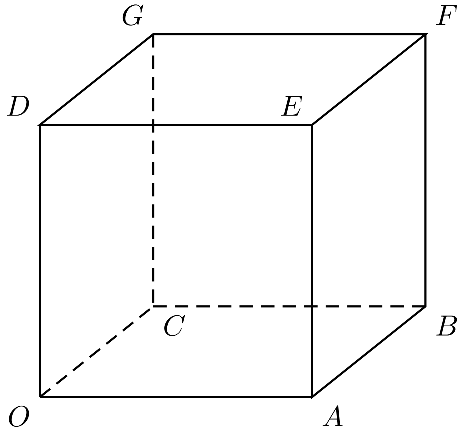


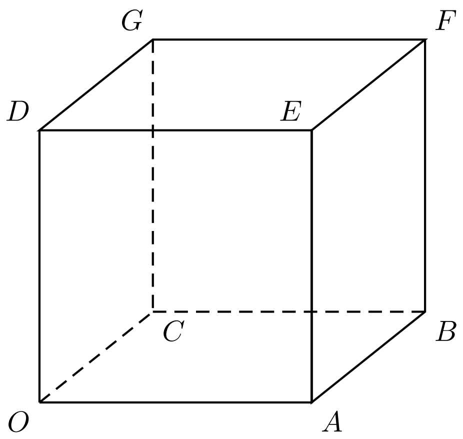








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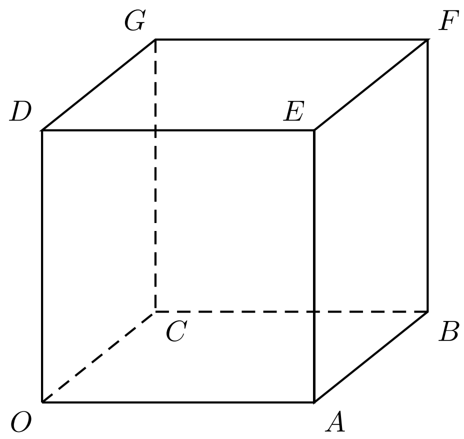
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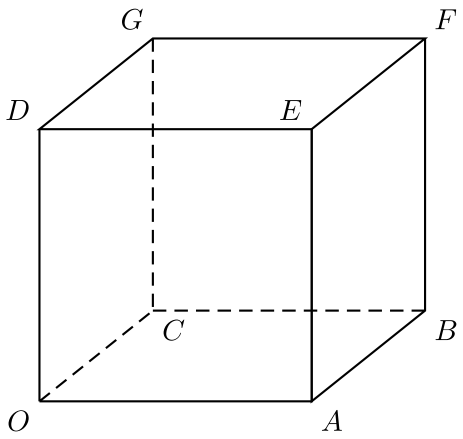
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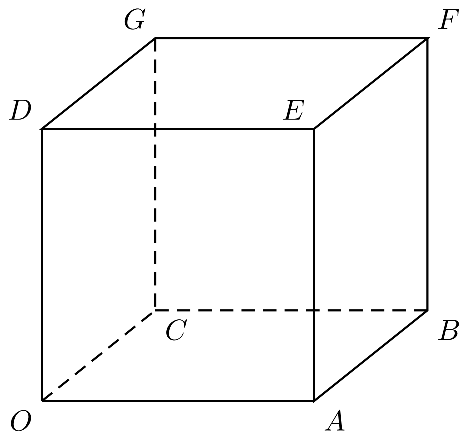
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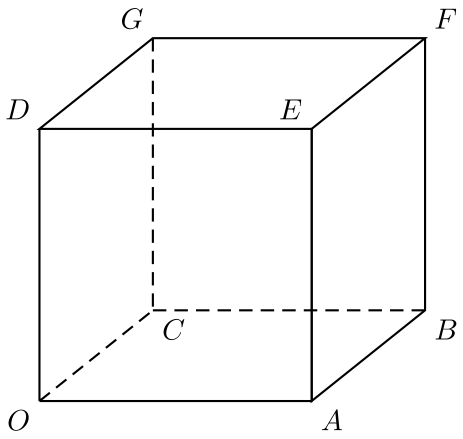
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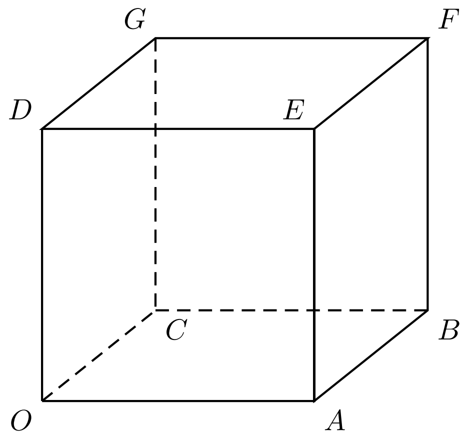
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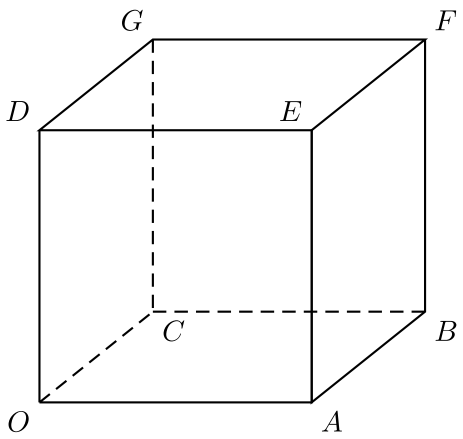
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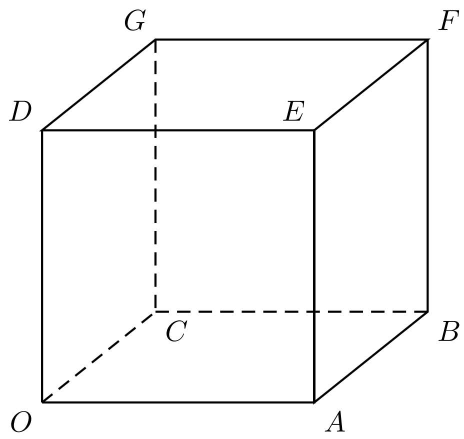
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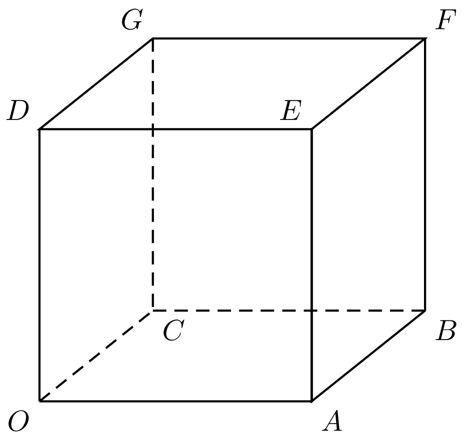
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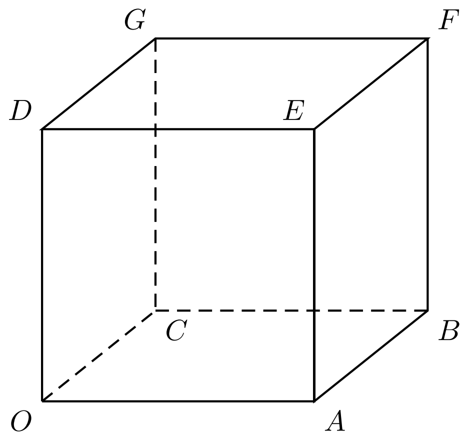
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1 Cartesian equations and planes

Definition 1

The set of points $M(x, y, z)$ in the 3D space such that

$$ax + by + cz = d$$

where a, b, c and d are real numbers is a *plane*.

Definition 2

The previous equality, true only for the points on the plane, is a *cartesian equation* of the plane.

Theorem 1

A plane has infinitely many different cartesian equations.

How to find a cartesian equation

> Find a cartesian equation of the plane $\mathscr{P}$ passing through the points of coordinates $P(1,0,0)$, $Q(\frac{1}{2},1,0)$ and $R(0,0,1)$.

We know that a cartesian equation has the form

$$ax + by + cz = d.$$

As the points P, Q, R are on the plane, we deduce that a, b, c and d are solutions to the system

$$\begin{cases} a \times 1 + b \times 0 + c \times 0 &= d \\ a \times \frac{1}{2} + b \times 1 + c \times 0 &= d \\ a \times 0 + b \times 0 + c \times 1 &= d \end{cases}$$

From the first equation we get $a = d$, and from the last $c = d$. Then, the second equation is equivalent to $\frac{1}{2}d + b = d$, or $b = \frac{1}{2}d$.
As there are infininitely many cartesian equation, we can choose for d any number (except 0 in this case). Let's choose $d = 2$.

Then we deduce $a = 2$, $c = 2$ and $b = 1$, so a cartesian equation of the plane is

$$2x + y + 2z = 2.$$

2 Parametric equations and lines

Definition 3

The set of points $M(x, y, z)$ in the 3D space such that

$$\begin{cases} x = x_A + at \\ y = y_A + bt \\ z = z_A + ct \end{cases}$$

where x_A, y_A, z_A, a, b and c are real numbers, is a *line* passing through point A and with directing vector $\vec{u}(a, b, c)$.

Definition 4

The previous system, true only for the points on the line, is a set of *parametric equations* of the line.

Theorem 2

A line has infinitely many different sets of parametric equations.

How to find parametric equations

Find a set of parametric equations of the line $\mathscr{L}$ passing through the point of coordinates $T(1, 1, 0)$ and with directing vector $\vec{u}(0, 0, 1)$.

According to the definition, a set of parametric equations of this line is

$$\begin{cases} x = 1 + 0 \times t \\ y = 1 + 0 \times t \\ z = 0 + 1 \times t \end{cases} \text{ or } \begin{cases} x = 1 \\ y = 1 \\ z = t \end{cases}$$

3 Intersections of lines and planes

Method

To find the coordinates of the intersection of a plane with a line (if it exists), just solve the system made of a cartesian equation of the plane and a set of parametric equations of the line.

- If the line intersects the plane in one just point, the solving will give one value for t, that can be used in the parametric equations to find the coordinates of the point.
- If the line is parallel to the plane, then it will be impossible to solve the system.
- If the line is included in the plane, then there will be an infinite number of solutions.

An example

Find the coordinates of the intersection of the plane $\mathscr{P}$ with the line $\mathscr{L}$, if it exists.
To do so, we have to solve the system :

$$\begin{cases} 2x+y+2z &= 2 \\ x &= 1 \\ y &= 1 \\ z &= t \end{cases}$$

We can replace x, y and z by their expressions as a function of t in the first equation. That gives :

$$\begin{aligned} 2\times 1+1+2t &= 2 \\ 3+2t &= 2 \\ 2t &= -1 \\ t &= -\frac{1}{2} \end{aligned}$$

Then, we use this value of t and the parametric equations to find the coordinates of the point :

$$\begin{cases} x &= 1 \\ y &= 1 \\ z &= -\frac{1}{2} \end{cases}$$

So the plane and the line intersection in the point of coordinates

$$\left(1,1,-\frac{1}{2}\right).$$