Episode 20 – Ramsey numbers

European section – Season 2

A party problem

You're organising a party. Some of the guests will know each other, while others won't. For the sake of simplicity, we admit that the relation of knowing each other is symmetric.
What is the least number of guest to invite so that at least m people will mutually know each other, or at least n people will be complete strangers ?

A very simple party problem

What is the least number of guest to invite so that at least 2 people will mutually know each other, or at least 2 people will be complete strangers ?

A simple enough party problem

What is the least number of guest to invite so that at least 3 people will mutually know each other, or at least 2 people will be complete strangers ?

A just as simple party problem

What is the least number of guest to invite so that at least 2 people will mutually know each other, or at least 3 people will be complete strangers ?

A not so simple party problem

What is the least number of guest to invite so that at least 3 people will mutually know each other, or at least 3 people will be complete strangers ?

A difficult party problem

What is the least number of guest to invite so that at least 4 people will mutually know each other, or at least 3 people will be complete strangers ?

Mathematical model

Turn this problem into a graph problem.

Complete graphs

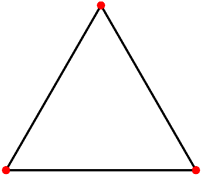
Complete graphs

K_2

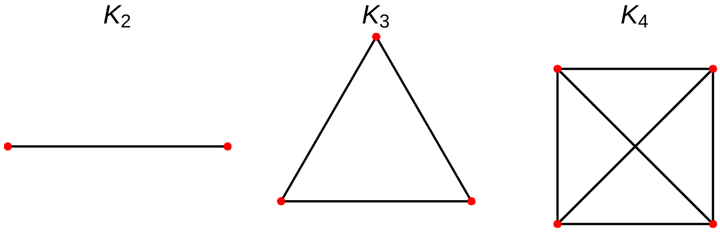
Complete graphs

K_2

K_3



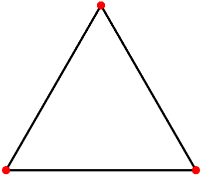
Complete graphs



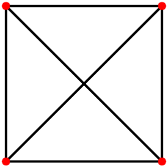
Complete graphs

K_2

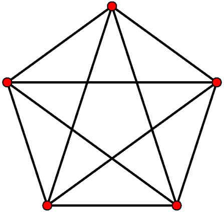
K_3



K_4



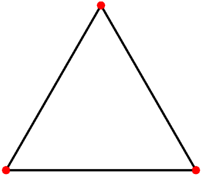
K_5



Complete graphs

K_2

K_3



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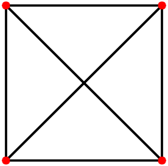
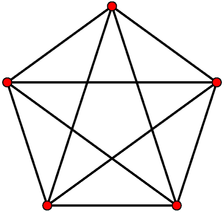
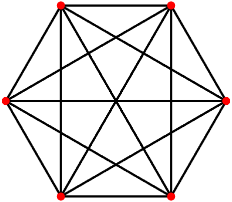


K_5



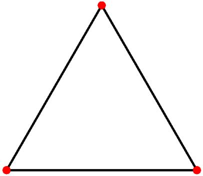
K_6



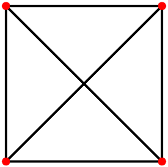
Complete graphs

K_2

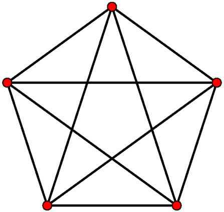
K_3



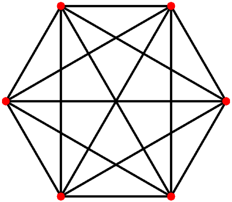
K_4



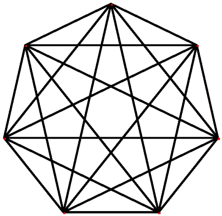
K_5



K_6



K_7



Frank P. Ramsey

Ramsey's problem

What is the lowest value of r such that when the edges of K_r are colored red or blue, there exists either a complete subgraph on m vertices which is entirely red, or a complete subgraph on n vertices which is entirely blue.

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This number is a *Ramsey number*, noted $R(m, n)$.

Some simple Ramsey numbers

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- $R(2,2) =$

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- $R(2,2) = 2$

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- $R(2,2) = 2$
- $R(3,2) =$

Some simple Ramsey numbers

- $R(2, 2) = 2$
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- $R(3,2) = 3$
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Some simple Ramsey numbers

- $R(2, 2) = 2$
- $R(3, 2) = 3$
- $R(m, 2) = m$

$R(3,3) \geq 5$

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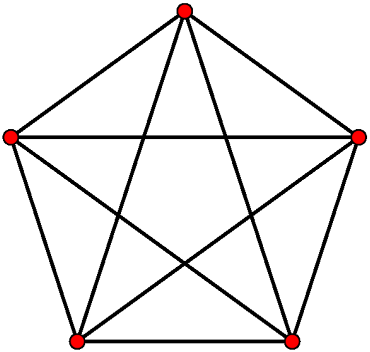
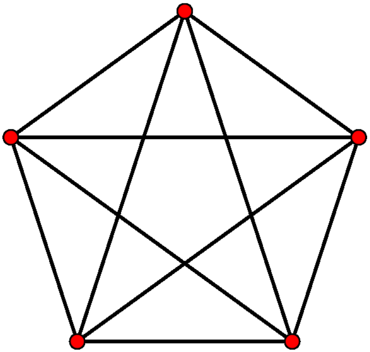
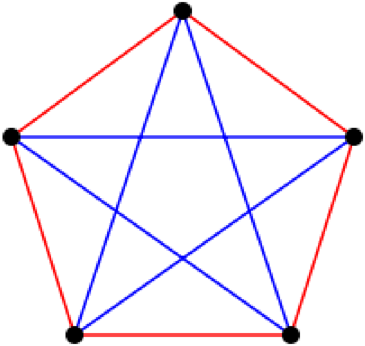
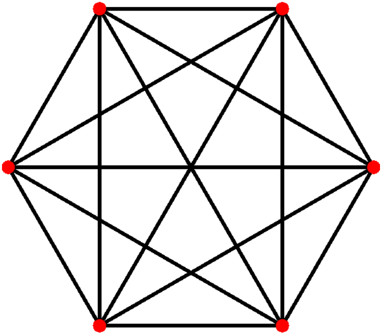




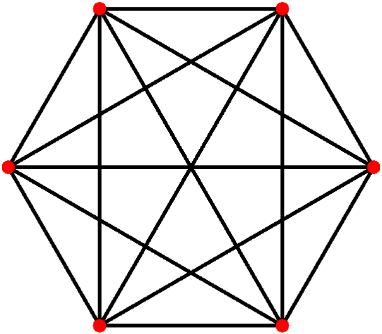


Suppose the edges of a complete graph on 6 vertices are coloured red and blue. Pick a vertex v.

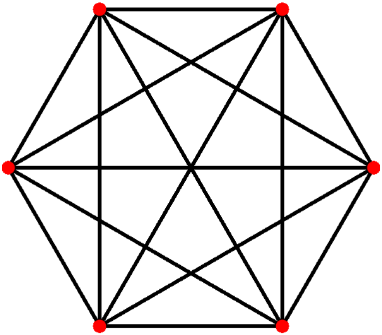


$R(3,3) = 6$

There are 5 edges incident to v and so (by the pigeonhole principle) at least 3 of them must be the same colour.

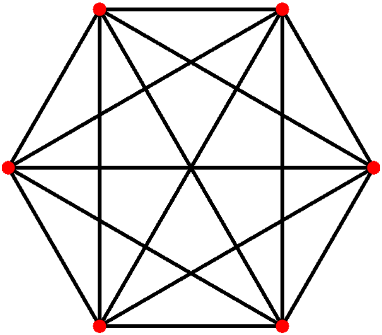


Without loss of generality we can assume at least 3 of these edges, connecting to vertices r, s and t, are blue. (If not, exchange red and blue in what follows.)



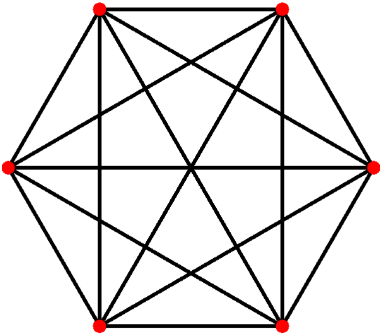
$R(3,3) = 6$

If any of the edges rs, rt, st are also blue then we have an entirely blue triangle.



$R(3,3) = 6$

If not, then those three edges are all red and we have an entirely red triangle.



What about $R(5,5)$ and $R(6,6)$?

Erdös asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5,5)$ or they will destroy our planet. In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value.

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But suppose, instead, that they ask for $R(6,6)$. In that case, he believes, we should attempt to destroy the aliens.