Planar graphs	Season	2
	Episode	21
	Time frame	2 periods

Prerequisites : Basic notions and vocabulary about graphs.

Objectives :
- Discover the concept of planar graphs.
- Use the main results about planar graphs to solve a few problems.

Materials :
- *Lesson.*
- *Problems.*
- *Beamer.*

1 – The three houses and three utilities problem — 10 mins

The problem of the three houses and three utilities is presented to the students. They have 10 minutes to try to solve it.

> Suppose we have three houses and three utility outlets (electricity, gas and water) situated as shown on the picture below. Is it possible to connect each utility to each of the three houses without the lines or mains crossing ?

2 – Planar graphs : a short introduction — 15 mins

The main notions and results about planar graphs are presented by the teacher with a beamer. Students are handed out a lesson at the end of the presentation

3 – Planar graphs : a few problems — Remaining time

Working by pairs or groups of three, students have to solve a few problems about planar graphs.

Planar graphs

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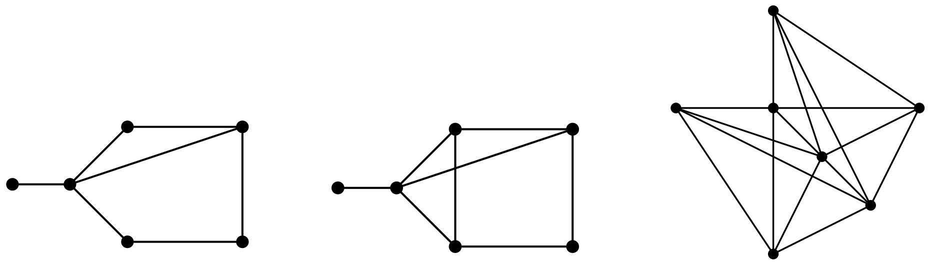
Document	Lesson

A *planar* graph is a graph that can be drawn in the plane in such a way that no two edges intersect except at a vertex. A *plane* graph is a graph that is actually drawn with no intersecting edges.

Plane Planar Non-planar



Let G be a planar graph. Its edges define *regions* of the plane. The vertices and edges that are incident to a region make up its *boundary*. For example, the planar graph shown as an example defines 3 regions of the plane : the outer region must always be counted.

When G is a planar graph, we usually denote p its number of vertices, q its number of edges and r its numbers of regions.

Theorem 1 Euler's formula

Let G be a connected plane graph with p vertices, q edges and r regions. Then

$$p - q + r = 2.$$

Theorem 2

Let G be a connected plane graph with p vertices, q edges, where $p \geqslant 3$. Then

$$q \leqslant 3p - 6.$$

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Problem 1

Show that the graphs of the tetrahedron, the cube and the octahedron are planar.

Problem 2

Draw all possible plane graphs with 1, 2 or 3 vertices.

Problem 3

Draw a non planar graph with 8 vertices.

Problem 4

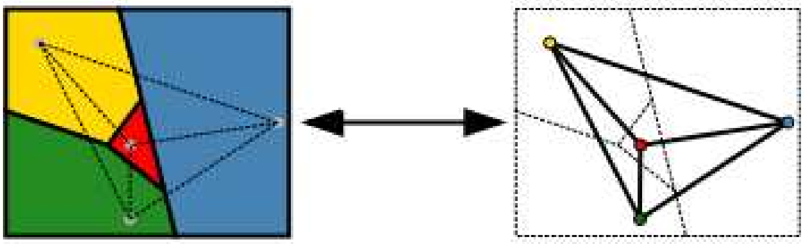
Use polydrons to build a dodecahedron and an icosahedron, then draw plane graphs for these two solids.

Problem 5

For what values of n is the complete graph K_n planar ? For each value of n such that it is so, draw K_n as a symmetric plane graph.

Problem 6

A graph can be obtained from any geographic map by replacing every region by a vertex, and connecting two vertices by an edge exactly when the two regions share a border segment (not just a corner).



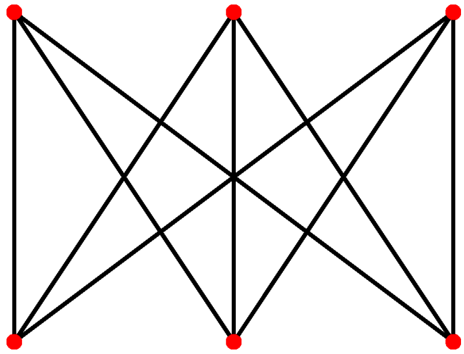
1. What special property exhibits any graph built in this way ?
2. Restate in the context of graph theory the famous Four color Theorem : "Given any separation of a plane into contiguous regions, called a map, the regions can be colored using at most four colors so that no two adjacent regions have the same color."

Problem 7

Use graph theory and planar graphs to re-state the three houses problem.

Problem 8

Let $K_{3,3}$ be the complete bipartite graph shown below. The aim of this exercise is to prove that $K(3,3)$ is not planar.



As usual, we call p the number of vertices of the graph, q its number of edges and r the number of regions defined by its edges. Moreover, we sum the number of edges lying on the boundary of each region, for all regions of $K(3,3)$, and denote this number N.

1. Give the value of p and q for $K_{3,3}$.
2. Suppose that $K_{3,3}$ is planar. Then, deduce from Euler's formula the value of r for any plane representation of this graph.
3. Prove that no representation of $K_{3,3}$ can contain a triangle. Deduce from this fact an inequality about N and r.
4. Explain why $N \leqslant 2q = 18$.
5. Deduce from the previous questions an upper bound for r.
6. Conclude.

Planar graphs	Season	2
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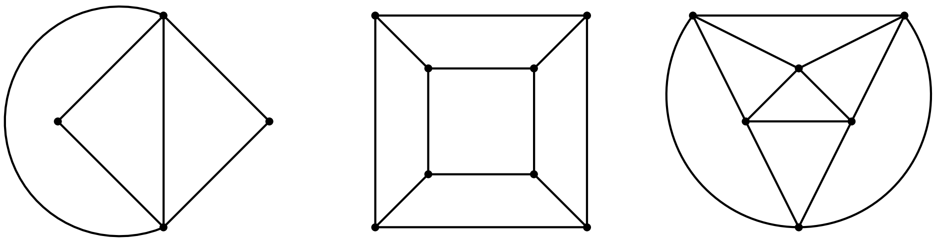
	Document	Solutions

Problem 1

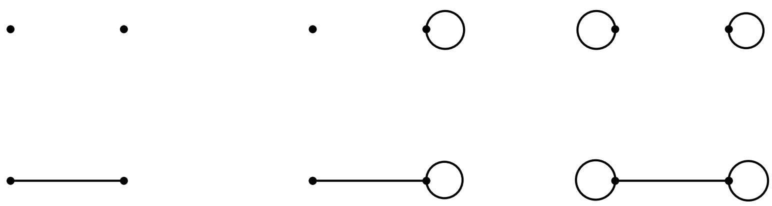
To show that the graphs of the tetrahedron, the cube and the octahedron are planar, we just have to draw them plane graphs of these three solids. These plane graphs are shown below.



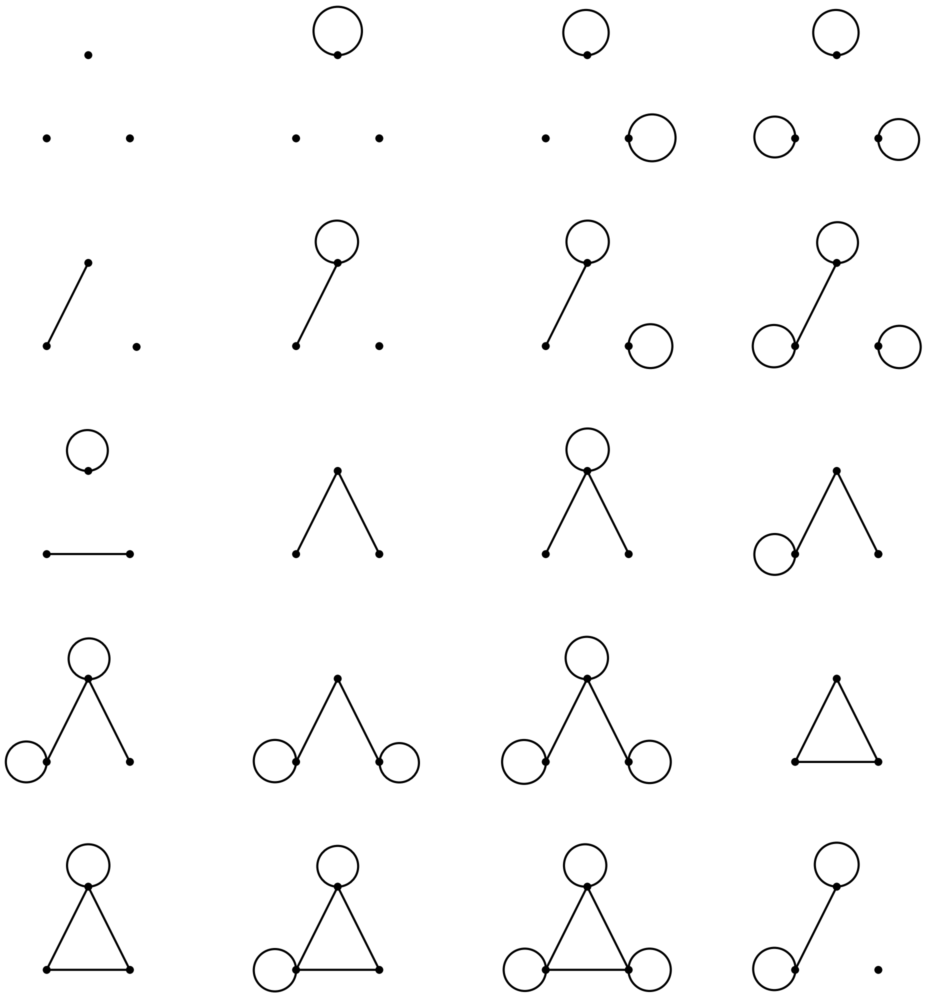
Problem 2

There are 2 plane graphs with 1 vertex :

There are 6 plane graphs with 2 vertices :

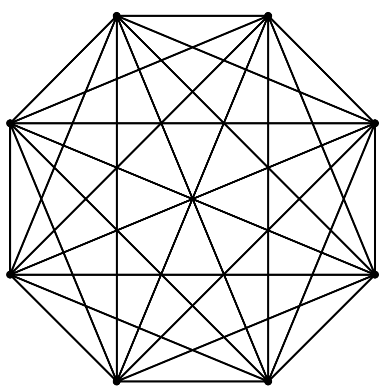


There are 20 plane graphs with 3 vertices :



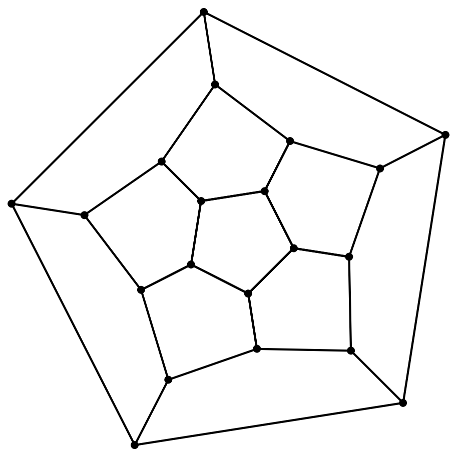
Problem 3

A simple example of non planar graph with 8 vertices is the complete graph K_8 :

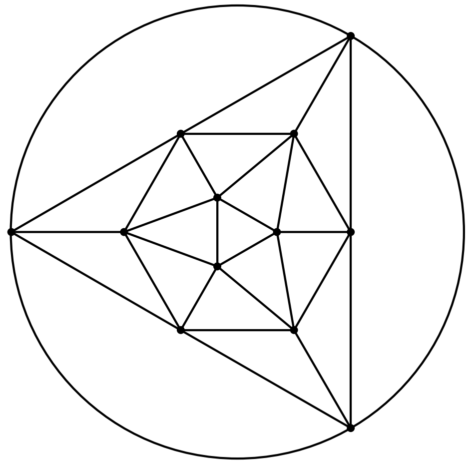


Problem 4

Plane graph of a dodecahedron :

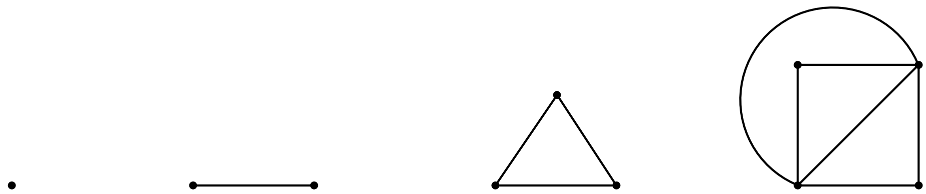


Plane graph of an icosahedron :



Problem 5

The complete graph K_n is planar only for $n \leqslant 4$. These graphs are shown below.



Problem 6

1. Any graph built in this way is plane. Indeed, as frontiers cannot cross each other on a map, edges cannot do it either in a graph built from a map.
2. In any plane graph, the vertices can can be colored using at most four colors so that no two adjacent vertices have the same color.

Problem 7

Consider a graph made of two groups of three vertices, such that each vertex of each group is adjacent to each vertex of the other group, but to no vertex of the same group. This graph is not planar.

Problem 8

1. For $K_{3,3}$, $p = 6$ and $q = 9$.
2. For any plane representation of $K_{3,3}$, we would still have $p = 6$ and $q = 9$. Then, according to Euler's formula, $p - q + r = 2$, so $6 - 9 + r = 2$, therefore $r = 5$.

3. First, we can notice that $K_{3,3}$ is made of two groups of vertices, each one with three vertices, and such that each vertex of each group is adjacent to each vertex of the other group, but to no vertex of the same group. Now, consider three vertices A, B, C in $K_{3,3}$. At least two of them must be in the same group. If it's exactly two, say A and B, then these two vertices are not adjacent, so A, B and C don't make a triangle. If the three vertices are in the same group, then none is adjacent to any other, so A, B and C don't make a triangle. Then, no representation of $K_{3,3}$ can contain a triangle.
This means that any region in such a plane graph will be bounded by at least 4 edges, so $N \geqslant 4r$.

4. Any edge can bound at most two regions, so $N \leqslant 2q$. But $q = 9$, so $N \leqslant 2q = 18$.

5. We know that $N \geqslant 4r$ and $N \leqslant 2q = 18$, so $4r \leqslant N \leqslant 18$, $4r \leqslant 18$, $r \leqslant 3.5$.

6. If question 2 we've seen that if $K_{3,3}$ is planar, then r must be equal to 5. But in question 5 we found that in any representation of $K_{3,3}$, $r \leqslant 3.5$. These two inequalities contradict each other, so $K_{3,3}$ is not planar.