Épreuve de section européenne

1 General knowledge

What kind of sequences do you know ? Give a few properties.

2 Document

The modern definition of a group is usually given in the following way.

Definition – A group G is a set with a binary operation $G \times G \to G$ which assigns to every ordered pair of elements x, y of G a unique third element of G (usually called the product of x and y) denoted by $x \times y$ such that the following four properties are satisfied :

Closure : if x, y are in G then $x \times y$ is in G.

Associative law : if x, y, z are in G then $x \times (y \times z) = (x \times y) \times z$.

Identity element : there is an element e in G with $e \times x = x \times e = x$ for all x in G.

Inverses : for every x in G there is an element u in G with $x \times u = u \times x = e$.

There is an embarrassing number of examples of groups. The most familiar ones come from elementary arithmetic.

- The integers form a group under the operation of addition. 0 is the identity and the inverse of an element is called its negative.
- Another common example of a group is the set of non-zero rational numbers with the group operation multiplication. In this group the inverse is called the reciprocal.
- A little thought convinces us that the positive rational numbers also form a group under multiplication.
- The set of negative rational numbers does not form a group under multiplication since it not only is not closed but also does not contain an identity, 1 nor inverses.
- Similarly the real numbers and the complex numbers are groups under addition and their non-zero elements form a group under multiplication.

These common examples are examples of infinite groups. There are many finite groups as well. In fact, finite groups are often more interesting than infinite groups. Consider the set $\{1, -1\}$ together with the operation multiplication. It forms a group with exactly two elements. It is closed, obeys the associative property, contains the identity and, in this case, each element is its own inverse. A slightly more interesting example is the set $\{1, -1, i, -i\}$ again with the operation of multiplication.

Adapted from various sources.

3 Questions

1. Explain why “The set of the integers forms a group under the operation of addition.”
2. Explain why “The set the non-zero rational numbers is a group under the operation of multiplication.”
3. “The set of negative rational numbers does not form a group under multiplication since it is not closed.” Give an example which proves this fact.
4. Does the set of the integers form a group under the operation of multiplication ?
5. Explain the example of the set $\{1, -1, i, -i\}$.