Épreuve de section européenne

Proof of the irrationality of e

The number e can be defined has the infinite sum $e = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}$.

This definition can be used to prove that e is irrational, meaning that it can't be written as a fraction $\frac{a}{b}$.

We will prove this result by contradiction. Suppose that there exist two positive integers a and b such that $e = \frac{a}{b}$. Consider the number

$$x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) = b! \sum_{n=b+1}^{\infty} \frac{1}{n!}.$$

We will first show that x is an integer, then show that x is less than 1 and positive. The contradiction will establish the irrationality of e.

- To see that x is an integer, note that

$$\begin{aligned}
x &= b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!} \\
&= a(b-1)! - \sum_{n=0}^{b} \frac{1 \cdot 2 \cdot 3 \cdots (n-1)(n)(n+1)\cdots(b-1)(b)}{1 \cdot 2 \cdot 3 \cdots (n-1)(n)} \\
&= a(b-1)! - \sum_{n=0}^{b} (n+1)(n+2)\cdots(b-1)(b).
\end{aligned}$$

  Clearly, every term of this sum is an integer. Then, so is the number x.

- From the second part of its definition it's clear that $0 < x$. Moreover,

$$\begin{aligned}
x &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \\
&< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots \\
&= \frac{1}{b} \\
&< 1.
\end{aligned}$$

  So $0 < x < 1$.

Since there does not exist a positive integer less than 1, we have reached a contradiction, and so e must be irrational.

Adapted from *Wikipedia.org*, the free encyclopedia.

Questions

1. What other irrational numbers do you know ?

2. Suppose that the radical $\sqrt{10}$ is rational. Then there exist two positive integers m and n such that $\sqrt{10} = \frac{m}{n}$.

- **a.** What is the relation between m^2 and n^2 ?
- **b.** What can you say about the number of zeros at the end of the square of any integer ?
- **c.** Find a contradiction.
- **d.** What can you conclude about $\sqrt{10}$?

3. What is the meaning of the notation "$n!$" ?

4. Summarize in your own words the proof of the irrationality of e.

5.
- **a.** Prove that for any positive integer n,

$$\sum_{i=1}^{n} \frac{1}{(b+1)^i} = \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots + \frac{1}{(b+1)^n} = \frac{1}{b}\left(1 - \frac{1}{(b+1)^n}\right).$$

- **b.** Deduce the limit of $\sum_{i=1}^{n} \frac{1}{(b+1)^i}$ when n approaches $+\infty$.
- **c.** Where is this property used in the proof of the irrationality of e ?