Épreuve de section européenne

A proof by contradiction

There is a formula for solving the general cubic equation $ax^3 + bx^2 + cx + d = 0$, that is more complicated than the quadratic equation. But in this example, we wish to prove there is no rational root to a particular cubic equation without using the general cubic formula.

Theorem. There is no rational number solution to the equation $x^3 + x + 1 = 0$.

Proof. Assume to the contrary there is a rational number $\frac{p}{q}$, in reduced form, with p not equal to zero, that satisfies the equation. Then, we have $(\frac{p}{q})^3 + \frac{p}{q} + 1 = 0$.

After multiplying each side of the equation by q^3, we get the equation $p^3 + pq^2 + q^3 = 0$.

There are three cases to consider :

1. If p and q are both odd, then the left hand side of the above equation is odd. But zero is not odd, which leaves us with a contradiction.
2. If p is even and q is odd, then the left hand side is odd, again a contradiction.
3. If p is odd and q is even, we get the same contradiction.

The fourth case (p even and q even) is not possible because we assumed that is in reduced form. This completes the proof.

Adapted from *zimmer.csufresno.edu*

Questions

1. What is the general form of a quadratic equation ?
2. What is the definition of a rational number ?
3. Explain the expression "in reduced form".
4. Why can we say that "p [is] not equal to zero".
5. Why are we allowed to "multiply each side of the equation by q^3" ?
6. If p and q are both odd, explain why the left hand side of the above equation is odd.
7. Explain the other two cases.
8. Why is the fourth case said to be impossible ?
9. Do you know any other property which can be proved by contradiction ?