Épreuve de section européenne

Ordered sets

R is a (binary) relation on a set A, if, for each pair (a, b) of elements of A, the statement aRb has a meaning. We say that a is related to b, and we write aRb if the statement is true. For example the statement "a is the mother of b" describes a binary relation on the set A of living people.

- A *reflexive* relation R on set A is one where for all a in A, a is R-related to itself, that is to say for any a in A, aRa.
- A binary relation R on a set A is *antisymmetric* if, for all a and b in A, if a is related to b and b is related to a, then $a = b$.
- A binary relation R over a set A is *transitive* if it holds for all a, b, and c in A, that if a is related to b and b is related to c, then a is related to c, i.e. if aRb and bRc then aRc.

A *(partial) order* is a binary relation R over a set A which is reflexive, antisymmetric, and transitive.

A relation's property of "totality" can be described this way : that any pair of elements in the set are mutually comparable under the relation. In this case, the order is said to be total.

Examples :

1. The set of natural numbers equipped with its natural ordering (the less than or equal to relation) is an ordered set. This partial ordering is a total ordering.
2. The set of integers equipped with its natural ordering. This partial ordering is also total.
3. Another example is given by the divisibility relation on the set of the natural integers. For two natural numbers n and m, we write nRm if n divides m (without remainder). One easily sees that this yields a partial order.
4. The set of subsets of a given set ordered by inclusion. For instance the set of subsets of $\{x, y, z\}$ (which is $\{\emptyset, \{x\}; \{y\}; \{z\}; \{x, y\}; \{x, z\}; \{y, z\}; \{x, y, z\}\}$) is ordered by inclusion.

Adapted from various sources.

Questions

1. Prove that the relation "$\leqslant$" is reflexive, antisymmetric, and transitive on the set of the natural integers. Why is this a total ordering ?
2. Is the relation "is the mother of" reflexive ? Is it antisymmetric ? Is it transitive on the set of living people ?
3. In the set of the natural integers, what does "n divides m" mean ? Prove that the divisibility relation defines an order on the set of the natural integers.
4. In the example of the set of the subsets of $\{x, y, z\}$, can you compare $\{x, y\}$ and $\{y\}$? Can you compare $\{x\}$ and $\{y\}$? Is this order a total order ? Explain why the set of subsets of a given set is ordered by inclusion.