Épreuve de section européenne

The Rubick's Cube

Rubik's Cube is a mechanical puzzle invented in 1974 by the Hungarian sculptor and professor of architecture Erno Rubik. The $3 \times 3 \times 3$ version, which is the version usually meant by the term "Rubik's Cube", has nine square faces on each side. Typically, the faces of the Cube are covered by stickers in six solid colors, one for each side of the Cube. When the puzzle is solved, each side of the Cube is a solid color. It can have $(8! \times 3^{8-1}) \times (12! \times 2^{12-1})/2 = 43,252,003,274,489,856,000$ different positions (permutations) but the puzzle is advertised as having only "billions" of positions, due to the general incomprehensibility of such a large number to laymen. Despite the vast number of positions, all Cubes can be solved in twenty-seven or fewer moves.

The symmetric group on a set X, denoted by S_X or $Sym(X)$, is the group whose underlying set is the set of all bijective functions from X to X, in which the group operation is that of composition of functions, i.e., two such functions f and g can be composed to yield a new bijective function $f \circ g$. Using this operation, S_X forms a group.

The Rubik's Cube is a remarkable object because it provides a tangible representation of a mathematical group. The Rubik's Cube group can be thought of as the set of all cube operations with composition as the group operation. Any set of operations which returns the cube to the solved state, from the solved state, should be thought of as the identity transformation (the operation that does nothing). Any set of operations which solves the cube from a scrambled state should be thought of as an inverse transformation of the given scrambled state, since it returns the identity transformation.

Note : a function f from a set X to a set Y is said to be bijective if for every y in Y there is exactly one x in X such that $f(x) = y$.

Adapted from *various sources*

Questions

1. How many square faces (facets) does the $3 \times 3 \times 3$ version of the Cube have ?
2. How many cube units are needed to form the Cube ?
3. What is the aim of the puzzle ?
4. What does the symbol " !" in the expression $(8! \times 3^{8-1}) \times (12! \times 2^{12-1})/2$ mean ?
5. How do you define the composition $f \circ g$ of the functions f and g ?
6. What is a bijective function ? Can you give an example ? Can you state a theorem which is used to prove that a real valued function is a bijection ?
7. How can you define the "inverse" of a bijection ? Do you know any examples ?
8. What is the link between the Rubik's cube and the theory of groups in mathematics ?