Épreuve de section européenne

Golden rabbits

The earliest mathematical model of population growth can be found in the work of Leonardo of Pisa, in 1220. [...] It was about the reproductive behaviour of rabbits. Not in its biological sense, but numerological. Leonardo took as the basic unit a pair of rabbits – a natural enough hypothesis. Assume that in the beginning there is one pair of immature rabbits. These mature for a season. Every season after, they beget one immature pair, which in turn matures for a season. And of course, all newly mature pairs beget[1] one immature pair per season as well. Suppose that rabbits and their procreative urges never die. How many pairs of rabbits will have been begotten after n seasons ?

Suppose there are M_n mature pairs and I_n immature pairs in season n. Then we start out in season 1 with $M_1 = 0$, $I_1 = 1$. The growth laws are :

$$I_{n+1} = M_n \text{ and } M_{n+1} = M_n + I_n.$$

From *Does God play dice ?* by Ian Stewart

Questions

1. What is the difference between an immature pair and a mature one ?

2. Explain the growth laws given at the end of the text.

3. In a table, compute the values of M_n and I_n for n from 1 to 8.

4. Let T_n be the total number of pairs of rabbits in season n. Compute the values of T_n for n from 1 to 8.

5. Prove that for any natural number n, $T_{n+2} = T_{n+1} + T_n$ and deduce that $\frac{T_{n+2}}{T_{n+1}} \times \frac{T_{n+1}}{T_n} = \frac{T_{n+1}}{T_n} + 1$.

6. We admit that the ratio $\frac{T_{n+1}}{T_n}$ approaches a positive real number φ when n approaches $+\infty$.

- **a.** Explain why we can say that φ is a solution of the equation $x^2 - x - 1 = 0$.
- **b.** Compute the exact value of φ.

[1] procreate