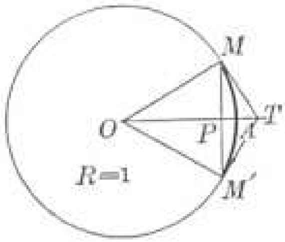
Épreuve de section européenne

The limit of $\frac{\sin x}{x}$ when x approaches 0



Let O be the center of a circle whose radius is unity.

Let arcAM = arc$AM' = x$, and let MT and $M'T$ be tangents drawn to the circle at M and M'. From Geometry, $MPM' < MAM' < MTM'$; or $2\sin x < 2x < 2\tan x$.

Dividing through by $2\sin x$, we get

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

If now x approaches the limit zero, $\lim\limits_{x\to 0} \frac{x}{\sin x}$ must lie between the constant 1 and $\lim\limits_{x\to 0} \frac{1}{\cos x}$, which is also 1.

Therefore $\lim\limits_{x\to 0} \frac{x}{\sin x} = 1$, or $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

From *Elements of the Differential and Integral Calculus* by William Anthony Granville

Questions

1. Explain the meaning of the notation MPM' used in this document.
2. Prove the inequalities $MPM' < MAM' < MTM'$ and explain how they induce $2\sin x < 2x < 2\tan x$.
3. Is it always correct to divide by $\sin x$?
4. Prove that $\lim\limits_{x\to 0} \frac{1}{\cos x} = 1$.
5. What is the main theorem used in this proof ?