Épreuve de section européenne

The irrationality of $\sqrt{3}$

The number $\sqrt{3}$ is irrational, ie., it cannot be expressed as a ratio of integers a and b. To prove that this statement is true, let us assume that $\sqrt{3}$ is rational so that we may write :

$$\sqrt{3} = \frac{a}{b} \tag{1}$$

for some integers a and b. We must then show that no two such integers can be found. We begin by squaring both sides of equation 1 :

$$3 = \frac{a^2}{b^2} \tag{2}$$

$$\text{or } 3b^2 = a^2 \tag{3}$$

If b is odd, then b^2 is odd ; in this case, a^2 and a are also odd. Similarly, if b is even, then b^2, a^2, and a are even. Since any choice of even values of a and b leads to a ratio $\frac{a}{b}$ that can be reduced by canceling a common factor of 2, we must assume that a and b are odd, and that the ratio $\frac{a}{b}$ is already reduced to smallest possible terms. With a and b both odd, we may write

$$a = 2m + 1 \tag{4}$$

$$\text{and } b = 2n + 1 \tag{5}$$

where we require m and n to be integers (to ensure integer values of a and b). When these expressions are substituted into equation 3, we obtain

$$3(4n^2 + 4n + 1) = 4m^2 + 4m + 1. \tag{6}$$

Upon performing some algebra, we acquire the further expression

$$2(3n^2 + 3n) + 1 = 2(m^2 + m). \tag{7}$$

The left hand side of equation 7 is an odd integer. The right hand side, on the other hand, is an even integer. There are no solutions for equation 7. Therefore, integer values of a and b which satisfy the relationship $\sqrt{3} = \frac{a}{b}$ cannot be found. We are forced to conclude that $\sqrt{3}$ is irrational.

From various sources.

Questions

1. What is the definition of $\sqrt{3}$?
2. What is the definition of an irrational number ?
3. What kind of proof do we use here ?
4. Explain why "if b is odd, then b^2 is odd".
5. Give the reason why "in this case, a^2 and a are also odd".
6. Explain the end of the proof.
7. Do you know any similar proof to show the irrationality of another real number ?