Épreuve de section européenne

Sums of Consecutive Numbers

Which numbers can be expressed as the sum of consecutive integers ?

It's easy to see that any multiple of 3, $3n$, is the sum of three consecutive integers.

Studying a few examples, it seems that a positive integer can be expressed as the sum of consecutive integers if and only if it's not a power of 2. We will prove this property.

Let us analyze what values can be taken by the sum of (two or more) consecutive positive integers from a to b ($b > a$)

$$S = a + (a+1) + (a+2) + \ldots + (b-1) + b = \left(\frac{a+b}{2}\right)(b-a+1).$$

Then, doubling both sides, we get :

$$2S = (a+b)(b-a+1).$$

Calling $a+b = x$ and $b-a+1 = y$, we can note that x and y are both integers and that since their sum, $x+y = 2b+1$, is odd, one of x, y is odd and the other is even. Note that $2S = xy$.

If S is not a power of 2, let $S = m2^n$, where m is an odd number greater than 1. We have $2m2^n = xy$, or $m2^{n+1} = xy$. We will now find positive integers a and b such that $b > a$ and $S = a + (a+1) + \ldots + b$. The two numbers 2^{n+1} and m are not equal, since one is odd and the other is even. Therefore, one is bigger than the other. Assign x to be the bigger one and y to be the smaller one. This assignment gives us a solution for a and b, as $x + y = 2b + 1$, giving a positive integer value for b, and $x - y = 2a - 1$, giving a positive integer value for a. Also, $y = b - a + 1 > 1$, so $b > a$, as required. We have obtained a and b.

Therefore, for any S that is not a power of 2, we can find positive integers a and b, $b > a$, such that $S = a + (a+1) + \ldots + b$.

Adapted from *Maths Wonders to Inspire Teachers and Students* by Alfred S. Posamentier

Questions

1. Try to write as a sum of at least two consecutive numbers all the integers from 10 to 16.
2. Show how any multiple of 3, $3n$, can be expressed the sum of three consecutive integers.
3. Explain the formula given for the sum of the consecutive integers from a to b.
4. Use the method given in the proof to express 30 as a sum of consecutive integers ? Is it the only possible decomposition ?
5. Prove that if S is a power of 2, it cannot be expressed as a sum of consecutive integers.