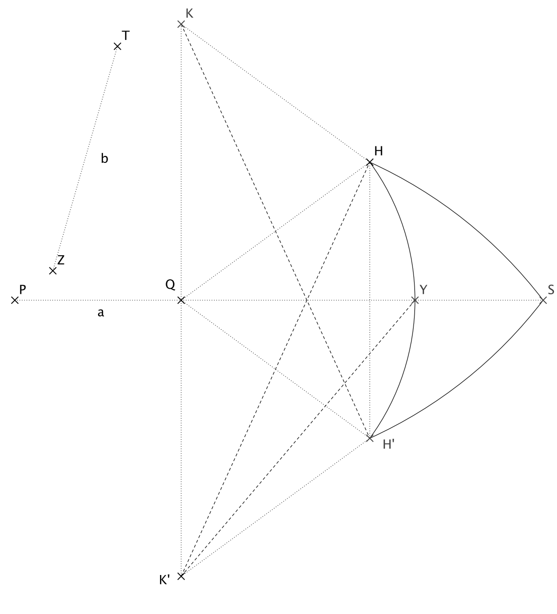
Épreuve de section européenne

A Mascheroni compass problem

In Mascheroni's geometry of the compass, all geometric constructions are executed with a compass alone, without the use of a straight-edge ; for example, the mirror image O' of a point O on a straight line AB is the point of intersection of the arc with center A and radius AO, and the arc with center B and radius BO.

Problem : To draw the sum of two given segments a and b.
(In other words : given P and Q $[PQ = a]$,
to draw Y collinear to P and Q such as $QY = ZT = b$.)

1. We draw the arc $Q|b$[1], take upon this arc any point H, draw the mirror image H' of H on the straight line determined by the points P and Q, and designate the segment HH' as h. We also define the point Y, intersection of the arc and the line PQ.
2. We draw the vertices K and K' of the isosceles trapezoid $KHH'K'$ whose legs KH and $K'H'$ are equal to b and whose base $HH' = 2h$. Let the diagonal $KH' = K'H$ of the trapezoid be called d. Since the trapezoid is a quadrilateral that can be inscribed in a circle, according to Ptolemy's theorem[2] the following equation is applicable : $d^2 = b^2 + 2h^2$.
   On the other hand, it follows from the right triangle $QK'Y$, that : $x^2 = b^2 + h^2$, where $K'Y$ is designated as x. From these two equations it follows that $d^2 = x^2 + h^2$ so that x is one of the legs of a right triangle with the hypotenuse d and the other leg h. If we then find the point of intersection S of the arcs $K|d$ and $K'|d$ on the straight line PQ, it follows that $QS = x$.



3. We draw the point of intersection of the arcs $K|x$ and $K'|x$; this is the point Y that we have been trying to find.

Adapted from *100 Great problems of elementary mathematics* by Heinrich Dörrie, Dover, 1965

Questions

1. How would you make this construction easier using compass *and* straight-edge ?
2. Explain Mascheroni's construction stressing the following points :
   - **a.** Explain the construction of points K and K'.
   - **b.** Give a detailed account of how Ptolemy's theorem is used in this construction.
   - **c.** Prove that S in indeed on PQ.

[1] Let arc $Q|b$ mean the circle arc whose center is Q and radius b (see figure).
[2] Ptolemy's theorem states that if $ABCD$ is a quadrilateral inscribed in a circle, then we have the equality : $AC \times BD = AB \times DC + AD \times BC$.