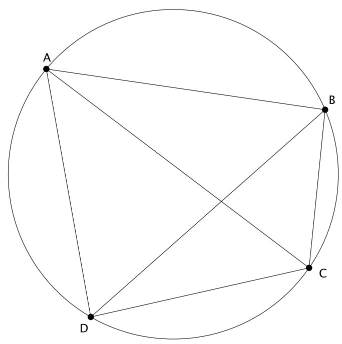
Épreuve de section européenne

Ptolemy's theorem

Ptolemy's theorem is a relation in Euclidean geometry between the four sides and two diagonals of a cyclic quadrilateral (a quadrilateral whose vertices lie on a common circle). The theorem is named after the Greek astronomer and mathematician Ptolemy.

If the quadrilateral is given with its four vertices A, B, C, and D in order, then the theorem states that :

$$AC \times BD = AB \times DC + AD \times BC.$$



Geometric proof

Let ABCD be a cyclic quadrilateral (see figure).

1. Note that on the chord BC, the inscribed angles $\angle BAC = \angle BDC$, and on AB, $\angle ADB = \angle ACB$.
2. Construct K on AC such that $\angle ABK = \angle CBD$.
3. Note that since $\angle ABK + \angle CBK = \angle ABC = \angle CBD + \angle ABD$, then $\angle CBK = \angle ABD$.
4. Now, by common angles $\triangle ABK$ is similar to $\triangle DBC$, and likewise $\triangle ABD \sim \triangle KBC$.
5. Thus $\dfrac{AK}{AB} = \dfrac{CD}{BD}$, and $\dfrac{CK}{BC} = \dfrac{DA}{BD}$; then $AK \times BD = AB \times CD$ and $CK \times BD = BC \times DA$. Adding, $(AK + CK) \times BD = AB \times CD + BC \times DA$; but $AK + CK = AC$, so $AC \times BD = AB \times CD + BC \times DA$.

Adapted from *Wikipedia*, the free encyclopedia

Questions

1. Who was Ptolemy ? How can his theorem be worded[1], starting with "In a cyclic quadrilateral, the product of. . ." ?
2. What is the meaning of notations $\angle$, $\triangle$, and $\sim$?
3. Explain each step of the geometric proof, especially justifying : the equality of angles in 1 ; the construction of K in 2 ; the similarity of triangles in 4 ; the final calculations in 5.
4. Why can you apply Ptolemy's theorem to a square ? To a rectangle ? What do you find as a result in each case ?
5. Let $ABCDE$ be a regular pentagon, which side and chord are named a and b respectively. Find the relationship between a and b. (The chords of the pentagon are AC, BD, CE, etc.)

[1] Worded : written in plain English.