Épreuve de section européenne

Different Computations of the square root of 2

Rough estimation : Many of the algorithms used for calculating the square root of a positive real number a require an initial value. It is useful to have a rough estimate, which may be very inaccurate but easy to calculate. If $a \geqslant 1$, let D be the number of digits to the left of the decimal point. Then the rough estimation is this :

- If D is odd, $D = 2n+1$, then use $\sqrt{a} \approx 2 \times 10^n$.
- If D is even, $D = 2n+2$, then use $\sqrt{a} \approx 6 \times 10^n$.

The Babylonian method : Perhaps the first algorithm used for approximating $\sqrt{a}$ is known as the "Babylonian method" (though there is no evidence of the use of such a method in Babylonian mathematics). This is a quadratically convergent algorithm, which means that the number of correct digits of the approximation roughly doubles with each iteration. It proceeds as follows :

1. Start with an arbitrary positive start value x_0 (for example the rough value of $\sqrt{a}$).
2. Let x_{n+1} be the average of x_n and $\frac{a}{x_n}$.
3. Repeat step 2 until the desired accuracy is achieved.

Example : To calculate $\sqrt{1253}$ to 3 significant decimals, we will use the rough estimation method above to get x_0. The number of digits in 1253 is $D = 4 = 2 \times 1 + 2$. So, $n = 1$ and the rough estimate is $x_0 = 6 \times 10^1 = 60.000$. Then $x_1 = \frac{1}{2} \times \left(x_0 + \frac{1253}{x_0}\right) = 40.442$; $x_2 = \frac{1}{2} \times \left(x_1 + \frac{1253}{x_1}\right) = 35.712$, etc.

The Bakhshali method : This is a method for finding an approximation to a square root which was described in an ancient manuscript (probably dating back to the 2nd century A.D.) known as the Bakhshali Manuscript. It is equivalent to two iterations of the Babylonian method beginning with the integer N, such as N^2 is the greatest perfect square less than a. Then, calculate :

$$d = a - N^2 \,;\; P = \frac{d}{2N} \,;\; A = P + N \text{ and finally } \sqrt{a} \approx A - \frac{P^2}{2A}.$$

Adapted from *Wikipedia*, the free encyclopedia.

Questions

1. By calculating the squares of 2×10^n and 6×10^n, justify the rough approximation method.
2. What is the rough estimation of $\sqrt{2}$?
3. What is a quadratically convergent algorithm ?
4. Translate step 2 of the Babylonian method into an algebraic formula.
5. Continue the calculations in the example of the Babylonian method to find the approximation of $\sqrt{1253}$ to the third decimal place.
6. Calculate an approximation of $\sqrt{2}$ to the fourth decimal place, using the Babylonian and Bakhshali methods.