Épreuve de section européenne

The trapezoidal rule

Many applications involve a definite integral $\int_a^b f(x)\mathrm{d}x$. We know how to calculate the value of such an integral if we can determine the antiderivative of $f(x)$, but certain simple functions (for instance e^x) have no antiderivative that can be expressed in terms of elementary functions. The trapezoidal rule is a method for numerically approximating the value of the integrals, that can be used even when $f(x)$ is not explicitly known but is only known for certain values of x, as is the case, for example, when $f(x)$ is obtained from a set of experimental data.

To begin, we subdivide the x-axis from $x = a$ to $x = b$ into n equal subintervals, each of length $\Delta x = (b - a)/n$. We denote the endpoints of the subintervals by a_0, a_1, a_2,…,a_n so $a_0 = a$, $a_1 = a + \Delta x$, $a_2 = a + 2\Delta x$,…, $a_n = b$. The Riemann sum approximation of the integral is obtained by choosing a certain value of $f(x)$ in each subinterval and taking this value as a constant value of f in this interval : it might be called the *rectangle rule*.

There is another way of approximating the integral for which computation is easier : on each subinterval, let us approximate $f(x)$ by the straight line connecting $(a_i, f(a_i))$ and $(a_{i+1}, f(a_{i+1}))$; the function $f(x)$ is then approximated by a broken line and the area under the curve is approximated by the area under the broken line.

To find the area under the broken line, we find the areas of each of the trapezoids that are defined by the x-axis, the straight lines seen above and the vertical lines $x = a$. We see that the area of each trapezoid is

$$\frac{f(a_i) + f(a_{i+1}}{2}(a_{i+1} - a_i) = \frac{f(a_i) + f(a_{i+1}}{2}\Delta x. \quad (1)$$

Summing up the areas of all trapezoids, we find the *trapezoidal rule* :

$$\int_a^b f(x)\mathrm{d}x \approx [f(a_0) + 2f(a_1) + 2f(a_2) + \ldots + 2f(a_{n-1}) + f(a_n)]\,\frac{\Delta x}{2}. \quad (2)$$

Adapted from *Calculus and its applications*
by Goldstein, Lay & Schneider, Prentice-Hall, NJ, 1980

Questions

1. In which case is the trapezoidal rule particularly useful ?

2. Justify the fact that Riemann approximation can be called the *rectangle rule* (you can draw a geometrical representation of the situation.)

3. Justify equation (1) and the name of the *trapezoidal rule*.

4. Prove relation (2).

5. Let's use the trapezoidal rule to approximate $\int_0^1 \frac{1}{1+x}\mathrm{d}x$.

- **a.** Take $n = 10$, so $\Delta x = \frac{1}{10}$. You get $a_0 = 0$, $a_1 = \frac{1}{10}$, $a_2 = \frac{2}{10}$, $a_3 = \frac{3}{10}$,…,$a_9 = \frac{9}{10}$, $a_{10} = 1$. Calculate $f(a_0)$, $f(a_1)$, $f(a_2)$, $f(a_3)$,…,$f(a_9)$, $f(a_{10})$ and show that the trapezoidal rule gives 0.69377 as a result.
- **b.** Find the exact value of the integral, using the antiderivative of $\frac{1}{1+x}$. Compare the two results.