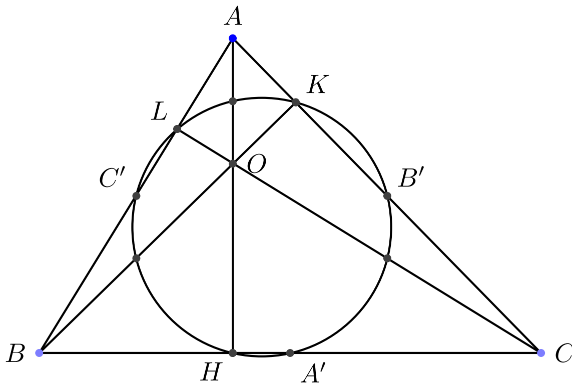
Épreuve de section européenne

The Feuerbach Circle

In every triangle, the three midpoints of the sides, the three base points of the altitudes, and the midpoints of the three altitude sections touching the vertices lie on a circle.



The proof consists of two steps : in the first we demonstrate that the circle circumscribing the triangle of the three midpoints of the sides passes through the base points of the altitudes ; and in the second we show that the circle circumscribing the triangle of the altitude base points passes through the midpoints of the altitude sections.

Step I. Let A', B' and C' represent the midpoints, respectively, of sides BC, AC and AB. Let H be the base point of the altitude AH. Then the trapezoid[1] $HA'B'C'$ is isosceles and it is therefore a quadrilateral inscribed in a circle, that we will name $\mathscr{C}$. In the same manner we would demonstrate the other altitudes base points, namely K and L, lie on circle $\mathscr{C}$, circumscribing triangle $A'B'C'$.

Step II. Let the altitudes of the triangle ABC be AH, BK, CL, and O their point of intersection. We will now show that the center of each altitude section touching a vertex, let us say section OC, also lies on circle $\mathscr{C}$. *[End of this proof to be completed in question 3.]*

Adapted from Heinrich Dörrie's *100 Great Problems of Elementary Mathematics*, Dover, 1965.

Questions

1. What are the "midpoints of the three altitude sections touching the vertices" ?
2. In step I, explain why $HA'B'C'$ is a trapezoid and why it is isosceles [Hint : HC' is the radius of the circle which has AB as its diameter.]
3. Completing step II :
   (a) Consider triangle OBC : what are its altitude bases ? What is their circumscribed circle ?
   (b) Apply the result of step I to triangle OBC : what is the circumcircle of these 3 points ?
4. Show that an isosceles trapezoid is always inscribed in a circle (you can sketch the situation ; it may be useful to consider the perpendicular bisectors of the two non-parallel sides.)

1. trapezoid (US) = trapezium (GB)