Épreuve de section européenne

Nixon's Quadrilateral

Three sides of a convex quadrilateral $ABCD$ have lengths $AB = a$, $BC = b$ and $DC = c$. If the area of the quadrilateral is as large as possible, the length x of the fourth side satisfies the equation :

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$

Proof : When $ABCD$ has maximum area, suppose the length of $AC = t$. Now in Figure 1,

$$\triangle ACD = \frac{1}{2}tc\sin\angle ACD \leqslant \frac{1}{2}tc\sin 90^\circ = \triangle ACD' \quad (1)$$

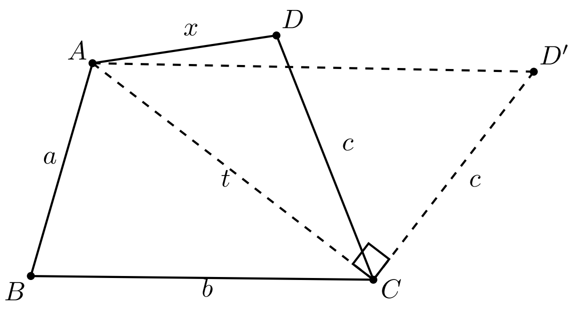

Figure 1

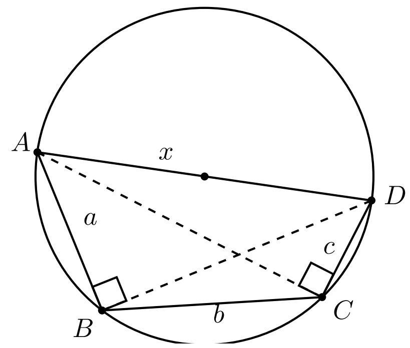

Figure 2

Therefore, if $\angle ACD$ is not a right angle, $\triangle ACD < \triangle ACD'$ and $ABCD'$ would be a quadrilateral with the prescribed dimensions which has an area greater than $ABCD$. Hence for maximum area, $\angle ACD$ must be a right angle, that is to say, the fourth side x must subtend a right angle at C. Similarly, x must subtend a right angle at B, and it follows that for maximum area, $ABCD$ must be inscribed in a circle of diameter $AD = x$.

Now by Ptolemy's theorem [admitted], we have : $AC \times BD = AB \times CD + BC \times AD$. From right triangles ACD and ABD, then, this yields :

$$\sqrt{x^2 - c^2} \times \sqrt{x^2 - a^2} = ac + bx \quad (2)$$

[Two hidden lines]

$$x^4 - (a^2 + b^2 + c^2)x^2 - 2abcx = 0 \quad (3)$$

And since $x \neq 0$, we obtain the desired $x^3 - (a^2 + b^2 + c^2)x - 2abc = 0$.

Adapted from Ross Honsberger, *Mathematical Chestnuts From Around the World*, MAA, 2001.

Questions

1. What is the meaning of symbols $\angle$ and $\triangle$?
2. (Série S students) Which well-known formula is used line (1) ?
3. What does Ptolemy's theorem state ?
4. Explain equality (2)
5. Find the content of the two hidden lines in the final demonstration.
6. Supposing $a = b = c = 1$, prove that x must equal 2, and make a sketch.