Épreuve de section européenne

Gauss optimization for complex multiplication

A complex number is an expression of the form $a+bi$, where a and b are real numbers, and i has the property that $i^2=-1$. The real number a is called the real part of the complex number, and the real number b is the imaginary part. When the imaginary part b is 0, the complex number is just the real number a.

Suppose that you want to multiply two complex numbers, $a+bi$ and $c+di$. To do so, you use the following rule :

$$(a+bi)(c+di)=[ac-bd]+[ad+bc]i.$$

Expressed in terms of a program, you would input a, b, c, and d and output $ac-bd$ and $ad+bc$.

Computationally, multiplying two digits is much more costly than adding two digits. Suppose then that multiplying two real numbers costs \$1 and adding them costs a penny. To obtain $ac-bd$ and $ad+bc$ requires four multiplications and two additions, for a total of \$4.02.

Is there a cheaper way to obtain the output from the input ? The Gauss optimization algorithm offers an alternative approach. Here's how the computation can be done for \$3.05, with three multiplications and five additions.

$$\begin{aligned} x_1 &= a+b & x_4 &= ac & x_6 &= ac-bd \\ x_2 &= c+d & x_5 &= bd & x_7 &= x_3-x_4-x_5 \\ x_3 &= x_1x_2 = ac+ad+bc+bd & x_6 &= x_4-x_5 & x_7 &= bc+ad \end{aligned}$$

So, Gauss optimization saves one multiplication out of four.

Adapted from *Divide-and-Conquer Multiplication* by Ivars Peterson

Questions

1. Check the usual complex multiplication rule is consistant with the definition of i.
2. Carry out the multiplication $(2+3i)(1+2i)$ with the usual method.
3. Explain the cost of the usual multiplication method.
4. Apply Gauss optimization to compute the product $(2+3i)(1+2i)$.
5. Explain the cost of the Gauss optimization algorithm.
6. Prove that the Gauss optimization algorithm is correct.