Épreuve de section européenne

A simple sequence not given by a simple formula

Let a_1, a_2, a_3,... be real numbers generated by the following rule. The first number, a_1, equals 1, and thereafter each number equals the previous number plus its square root. This simply stated rule raises an obvious question : what is the number a_n ?

To get a feel for the question, let us work out a_n for a few small values of n. We have $a_2 = 1 + \sqrt{1} = 2$. Then

$$\begin{aligned} a_3 &= 2 + \sqrt{2} \\ a_4 &= 2 + \sqrt{2} + \sqrt{2 + \sqrt{2}} \end{aligned}$$

and so on. Notice that the expressions on the right-hand side do not seem to simplify, and that each new one is twice as long as the previous one. From this observation it follows quite easily that the expression for a_{12} would involve 1024 occurences of the number 2, most of them inside a jungle of square roots. Such an expression would not give us much insight into the number a_{12}.

Adapted from *Mathematics, A Very Short Introduction* by Timothy Gowers

Questions

1. Translate the definition of the sequence a_n as a relation between a_{n+1} and a_n.
2. Give the raw formula for a_5.
3. Explain the sentence "Notice that the expressions on the right-hand side do not seem to simplify, and that each new one is twice as long as the previous one".
4. Prove that the expression of a_{12} would involve 1024 occurences of the number 2.
5. The aim of the following questions is to count the number of square root signs in a_n. Let's denote this number by u_n, for any natural number n.
   a. Find out the values of u_n for n from 1 to 5.
   b. Prove by induction that $u_n = 2^{n-1} - 1$.
   c. Use the formula to compute the number of square root signs in a_{12}.