Épreuve de section européenne

The irrationality of the Golden Ratio

The Golden Ratio is the ratio of the side lengths of a rectangle with the following property : if you cut a square out of it then you are left with a smaller, rotated rectangle of exactly the same shape as the original one. In this article we admit that such a ratio exists, and we prove that it is not a rational number.

Let us take a rectangle with sides of length x and 1, with $x > 1$, and consider the following process. First, cut off a square from it, leaving a smaller rectangle which, by the definition of the golden ratio, has the same shape as the original one. Now repeat this basic operation over and over again, obtaining a sequence of smaller and smaller rectangles, each of the same shape as the one before and hence each with side lengths in the Golden Ratio. Clearly, the process will never end.

Now let us do the same to a rectangle with side lengths in the ratio $\frac{p}{q}$ where p and q are whole numbers. This means that the rectangle has the same shape as a rectangle with side lengths p and q, so it can be divided into $p \times q$ little squares. What happens if we remove large squares from the end of this rectangle ? If q is smaller than p, then we will remove a $q \times q$ square and end up with a $q \times (p - q)$ rectangle. We can then remove a further square, and so on. Can the process go on forever ? No, because each time we cut off a square we remove a whole number of the little squares, and we cannot possibly do this more than $p \times q$ times because there were only $p \times q$ little squares to start with.

It follows that the ratio $\frac{p}{q}$ is not the Golden Ratio, whatever the values of p and q. In other words, the Golden Ratio is irrational.

Adapted from *Mathematics, A Very Short Introduction* by Timothy Gowers

Questions

1. What is the definition of the Golden Ratio ?

2. What is an irrational number ?

3. **a.** Consider a golden rectangle with sides 1 and x. Prove that x must satisfy the equality

$$\frac{x}{1} = \frac{1}{x - 1}.$$

b. Turn the previous equality into a quadratic equation and deduce the value of the Golden Ratio, knowing that it's a number greater than 1.

4. Draw two pictures to illustrate and explain in details the two parts of the proof in the text.