Épreuve de section européenne

Help Lewis Carroll to find RATs

On December 19, 1898, Lewis Carroll wrote in his diary :

Sat up last night till 4 a.m., over a tempting problem, sent me from New York, "to find three equal rational-sided right-angled triangles". (Abbreviation : RAT)[1] *I found two, whose sides are 20, 21, 29 and 12, 35, 37; but could not find three.*

Henry Dudeney answered in 1908 in his book *The Canterbury puzzles* (problem 107) :

Every reader should know that if we take any two numbers, m *and* n *then* $m^2 + n^2$, $m^2 - n^2$ *and* $2mn$ *will be three sides of a right angle triangle. Here* m *and* n *are called generators.*

To form three such triangles of equal area, we use the following simple formulae, where m *is the greater number :* $a = m^2 + n^2 + mn$; $b = m^2 - n^2$; $c = n^2 + 2mn$.

Now, if we form three triangles from the following generators : a *and* b, a *and* c, a *and* $b + c$; *they will all be of equal area.*

The following is a subtle formula by means of which we may always find a RAT equal in area to any given RAT :

Let z *be the hypotenuse,* B *the base,* h *the height and* A *the area of the given triangle, then all we have to do is to form a RAT from the generators* z^2 *and* $4A$, *and finally give each side*[2] *the denominator* $2z(B^2 - h^2)$, *and we get the required answer in fractions.*

From *The Universe in a Handkerchief* by Martin Gardner
and *The Canterbury puzzles*, by Henry Ernest Dudeney

Questions

1. **a.** Prove that if we take any two numbers m and n, then $m^2 + n^2$, $m^2 - n^2$ and $2mn$ and $2mn$ will be three sides of a rightangle triangle.

   b. Lewis Carroll's triangle 20, 21, 29 is a $m^2 + n^2$, $m^2 - n^2$ and $2mn$ RAT. For which values of m and n ?

2. **a.** Choose two values for m and n, and using Dudeney's first method, find the corresponding a, b and c. Deduce three right angled triangles from the pairs of generators : a and b, a and c, a and $b + c$.

   b. Prove that the triangles you found in question 2a are equal in area.

3. As your previous triangles do not have the same area as triangles 20, 21, 29, and 12, 35, 37 use the last method presented by Dudeney to find a rational-sided right-angled triangle equal in area to triangles 20, 21, 29, and 12, 35, 37.

1. equal triangles : here, with equal area
2. give each side the denominator 2z(B2-h2) : divide each side by the denominator 2z(B2-h2)