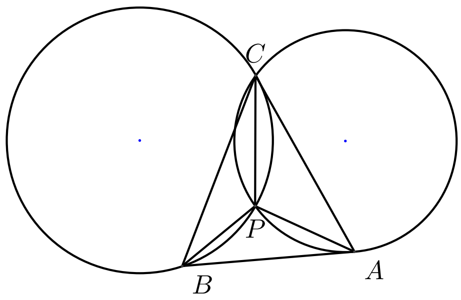
Épreuve de section européenne

Brocard points

We will prove that inside any triangle ABC, there exists a unique point P (figure 1) such that $\angle PAB = \angle PBC = \angle PCA$. This point is one of the two Brocard points of triangle ABC.



Proof : Indeed, if $\angle PAB = \angle PCA$, then the circumcircle of triangle ACP is tangent to the line AB at A. If S is the center of this circle, then S lies on the perpendicular bisector of segment AC, and the line SA is perpendicular to the line AB. Hence, this center can be constructed easily.

Therefore, point P lies on the circle centered at S with radius SA. (Note : this circle is not tangent to line BC unless $BA = BC$).

We can use the equation $\angle PBC = \angle PCA$ to construct the circle passing through B and tangent to line AC at C.

The Brocard point P must lie on both circles and be different from C. Such a point is unique. The third equation $\angle PAB = \angle PBC$ clearly holds.

We can construct the other Brocard point Q in a similar fashion, but in reverse order. We can also reflect lines AP, BP, and CP across the angle bisectors of $\angle CAB$, $\angle ABC$, and $\angle BCA$, respectively (figure 2).

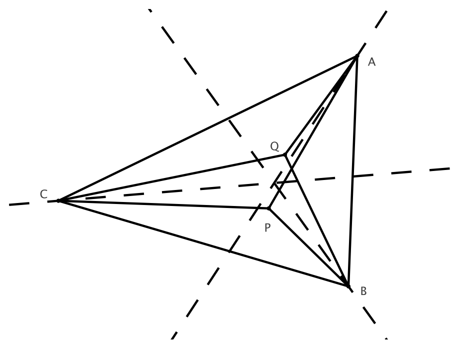


Figure 1

103 Trigonometry Problems, from the *Training of the USA IMO Team*
By Titu Andreescu and Zuming Feng

Questions

1. First of all, we are going to explain the first sentence of the proof :

- **a.** Prove, by working in triangle APS, that $\angle BAP + \angle PAS = 90°$. (You may use the "well-known" relation $\angle PSA = 2 \times \angle PCA$.)
- **b.** Explain the note in the second paragraph of the proof.

2.

- **a.** Draw a scalene triangle ABC, construct point S and the circle whose centre is S.
- **b.** Continue the construction of the point P then justify the fact that the equation $\angle PAB = \angle PBC$ holds and the statement about the other Brocard point Q.

3. Using reflections about the three angle bisectors AE, BF and CG of triangle ABC, explain the construction of Q, the second Brocard point. (see fig2)