Épreuve de section européenne

Richmond's construction of a regular pentagon

Equilateral triangles and squares are no trouble to draw with compass and straightedge alone, but most of us would be stuck if we were asked to draw a regular pentagon. However, it is not hard once you know how! This is how:

1. Take a circle with centre O; draw a radius OC perpendicular to a diameter AB.
2. Join B to D, the midpoint of OC.
3. The angle bisector of $\angle BDO$ intersects line BO at E.
4. Draw the perpendicular to BO passing through E; it cuts the circle at P and P'.

Then BP (or BP') is one side of the pentagon, and stepping that distance off around the circle gives the finished polygon.

Is this construction correct? Assume that the unit is radius OA. As $OD = \frac{1}{2}OB$, we have: $\tan 2\angle ODE = 2$. From the formula:

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \tag{1}$$

by taking θ for $\angle ODE$, it follows that

$$2(1-\tan^2\theta) = 2\tan\theta. \tag{2}$$

Solving that equation gives:

$$\tan(\theta) = \frac{\sqrt{5}-1}{2}$$

so

$$OE = \frac{\tan\theta}{2} = \frac{\sqrt{5}-1}{4}, \text{ and that is } \cos 72°. \tag{3}$$

Adapted from Underwood Dudley's *Mathematical Cranks*, MAA, 1992.

Questions

1. How do you construct an equilateral triangle with ruler and compass?
2. Choose a circle and sketch the inscribed pentagon according to steps 1 to 4.
3. Justify: $\tan 2\angle ODE = 2$.
4. Prove formula (1), keeping in mind that $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and $\sin 2\theta = 2\sin\theta \times \cos\theta$.
5. Justify the given solution to equation (2) (which is a quadratic equation with unknown $\tan\theta$.)
6. Explain line (3). Why does it establish that the constructed polygon is a regular pentagon?