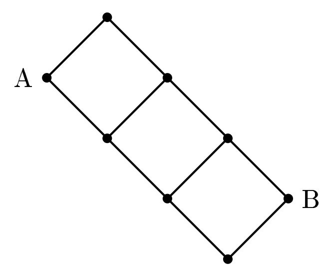
Épreuve de section européenne

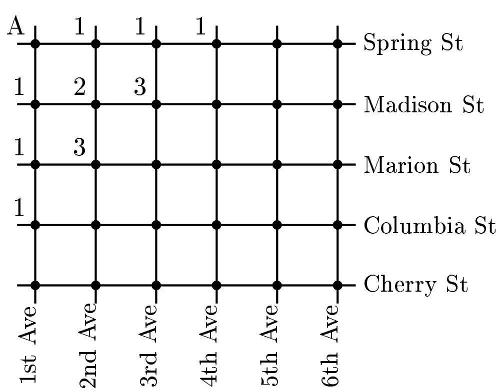
Pascal, Fibonacci and Seattle cabs.

Pascal's triangle is a pretty pattern of numbers usually taught in the secondary school. The triangle is shown below. To work out a number in the triangle, simply add the two above it (except the end numbers which are always 1).

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1



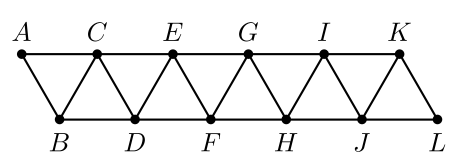
Pascal's triangle makes appearances in many real life situations, including the streets of Seattle. In a street system where the roads form a grid, a cab driver has a choice of routes which will reach a destination in the minimum distance. For example in the 3×1 grid above the cab wants to travel from A to B. It has four possible routes, each of which is the same length.



In fact the number of minimum distance routes available is always a number from Pascal's triangle, and the link becomes clear if you look at the routes available for a Seattle cabby driving from point A (where 1st Avenue meets Spring street) to any other junction (we are ignoring the existence of one-way streets here!). The choice of routes along rectangles leads us to Pascal's numbers, whereas routes along equilateral triangle would lead us to different numbers.

Indeed, imagine that a Seattle cab has to drive you from junction A along the triangular grid below, provided that: 1° the path doesn't have to be minimal anymore, and 2° the cab has to drive from left to right (no return). There is only one path to A (you don't move), one path to B ($A \to B$), two paths to C ($A \to C$ and $A \to B \to C$).

There are three paths to D, namely A→C→D, A→B→D and A→B→C→D, and so on. Going further, you get the Fibonacci number sequence.



Inspired by Rob Eastaway & Jeremy Wyndham, *Why do buses come in threes?* Portico, 2008.

Questions

1. Give the next line on Pascal's triangle. What is the computation algorithm?
2. What are the four possible routes mentioned in line 7?
3. Complete the Seattle street grid for each junction. Why do you get Pascal's numbers?
4. In the triangular grid, count the number of different paths from A to E, to F.
5. Conjecture the number of paths from A to G, to H.
6. The numbers found with the triangular grid are the numbers in the Fibonacci sequence: $F_1 = 1$, $F_2 = 1$ and for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Explain why this relation holds in general in this situation.