Épreuve de section européenne

The most efficient integer base.

Numerals in various bases may look different, but the numbers they represent are the same. In decimal notation, the numeral 19 is shorthand for this expression: $1 \times 10^1 + 9 \times 10^0$. Likewise the binary (or base-2) numeral 10011 is understood to mean: $1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$, which adds up to the same value. So does the ternary (that is: base-3) version, 201. The general formula for an integer in any positional notation goes like this: $d_k r^k + \ldots + d_3 r^3 + d_2 r^2 + d_1 r^1 + d_0 r^0$, where r is the base or radix and the coefficients d_i are the digits of the number. Usually, r is a positive integer and the digits are integers in the range from 0 to $r - 1$.

To say that all bases represent the same numbers, however, is not to say that all numeric representations are equally good for all purposes. The cultural preference for base 10 and the engineering advantages of base 2 have nothing to do with any intrinsic properties of the decimal and binary numbering systems. Base 3, on the other hand, does have a genuine mathematical distinction in its favor: by one plausible measure, it offers the most economical way of representing numbers.

In order to minimize the “cost” of writing numbers in a certain base, we need to optimize some joint measure of a number’s width (how many digits it has) and its depth (how many different symbols can occupy each digit position). An obvious strategy is to minimize the product of these two quantities. In other words, if r is the radix and w is the width in digits, we want to minimize rw while holding r^w constant. For example, let’s consider again the task of representing all numbers from 0 through decimal 999,999: in base 10 this obviously requires a width of six digits, so that $rw = 60$; binary does better: 20 binary digits suffice to cover the same range of numbers, so that $rw = 40$. But ternary is better: the ternary representation has a width of 13 digits, so that $rw = 39$.

If we want to solve the problem in general, it is easier to treat r and w as continuous rather than integer variables. Then it turns out that the optimum radix is e, the base of the natural logarithms, with a numerical value of about 2.718. Because 3 is the integer closest to e, it is almost always the most economical integer radix.

Adapted from Brian Hayes’s “Third-Base”, *Scientific American 89-6*, December 2001.

Questions

1. What is the difference between “numerals” and “numbers”?
2. Verify that the ternary number 201 means the same quantity as the decimal number 19. What would be its expression in base 4? In which base would it be expressed as 23?
3. Explain the expressions “cultural preference for base 10”; “engineering advantages of base 2”.
4. Explain the results for rw in the three examples given at the end of paragraph 3.
5. General proof: let’s consider $r^w = a$, where a is a constant positive real number. Using the natural logarithm ln, express w as a function of r. Study the variations of function $f : r \mapsto rw$ and deduce that e is the value of r that minimizes rw as it is asserted in the text.