Épreuve de section européenne

Perfect numbers

A perfect number is a natural number that is the sum of its proper divisors (all divisors except the number itself). For example, 28 is perfect. In The Elements, book IX, proposition 36, Euclid wrote:

> If as many numbers as we please, beginning from a unit, are set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last makes some number, the product will be perfect.

With the advantages of modern notation, we can express what Euclid meant more precisely: if we begin with 1 and add to it successively higher powers of 2 so the resulting sum $1+2+4+8+\ldots+2^n$ is a prime number, then the number $N_n = 2^n(1+2+4+8+\ldots+2^n)$, formed by multiplying the sum $1+2+4+8+\ldots+2^n$ by its "last" summand 2^n, must be perfect.

Note in passing that the numbers K_n of the form $1+2+4+8+\ldots+2^n$ don't need to be prime at all. Euclid's perfect number theorem applied only to those special cases where this sum indeed turns out to be a prime.

We shall not look at Euclid's proof of this result but shall instead consider a specific example. For instance, $1+2+4+8+16 = 31$, a prime. Then, $N = 16 \times 31 = 496$ should be perfect. To see that it is, we list all the proper divisors of 496 – namely, 1, 2, 4, 8, 16, 31, 62, 124 and 248 – and add them to get 496, as promised.

Adapted from *Journey through genius, the great theorems of mathematics*, by William Dunham

Questions

1. Use the text to explain the following words: prime number, proper divisor, perfect number.
2. Check that 28 is perfect.
3. Find a simpler formula for K_n and deduce the general expression of N_n.
4. In order to check that "numbers K_n of the form $1+2+4+8+\ldots+2^n$ don't need to be prime at all', complete a table of values of K_n, for n ranging from 1 to 5.
5. Check that N_3 is not perfect.
6. Proof of Euclid's theorem: We suppose that n is a natural number.
   (a) List all the divisors of 2^n.
   (b) Hence deduce all the proper divisors of $N_n = 2^n \times K_n$, where $K_n = 2^{n+1} - 1$ is prime.
   (c) Add up all the proper divisors found in the previous question and prove that N_n is perfect.