Épreuve de section européenne

Prime numbers

The primes have lost nothing of their fascination since they were first investigated well over two thousand years ago. Even Carl Friedrich Gauss, certainly one of the greatest mathematicians of all times, was drawn into their orbit. He wanted to know how the prime numbers are distributed among the integers. Can one say how many primes there are “far out there”?

Two facts are clear. First, the prime numbers seem to pop up like mushrooms without any particular rule, and secondly, it is clear that the larger the number, the less likely it is to be prime.

In 1792, based on his concrete calculations, Gauss conjectured what today is called the prime number theorem: the proportion of primes less than a given number N is almost exactly $\frac{0.43}{k}$ where k is the number of digits of N. Gauss was long dead when his conjecture was proved; the prime number theorem, which was proved in 1896 by Jacques Hadamard, now asserts that for any large x, the number of primes less than x is very well approximated by $\frac{x}{\ln x}$.

To appreciate the accuracy of the approximation we need to look at a few examples. For $x = 100000000$ there are 5761455 prime numbers less than x. The quotient differs by 332774, which represents an error of about six percent.

Since then, in the 20th century, much-more-refined descriptions of the distribution of prime numbers than that conjectured by Gauss, and proved by Hadamard, have appeared. Considering what is meant by “very well approximated” in the above discussion, a more careful analysis reveals that one can do much better with more-complicated closed formulas. For $x = 100000000$, the best one is off by only 754, with an error of about one-hundredth of one percent!

From Adapted from *Five-minute mathematics*, Ehrhard Behrends

Questions

1. How do you understand “prime numbers seem to pop up like mushrooms without any particular rule”?
2. How can it be that “the larger the number, the less likely it is to be prime”?
3. Using Gauss’s conjecture, evaluate the proportion of prime numbers less than 999. Hence find an approximate value of the number of primes less than 999.
4. Do the same to find an approximate value of the number of primes less than 999999, with Gauss’s formula.
5. Take $x = 10^k$, where k is a natural number, and show that Gauss’s conjecture was a very good approximation of Hadamard’s general formula.
6. Check the percentages given in the last two paragraphs of the text for $x = 100000000$.