Épreuve de section européenne

The Collatz conjecture

The Collatz conjecture is an unsolved conjecture in mathematics named after Lothar Collatz, who first proposed it in 1937. The conjecture is also known as the $3n+1$ conjecture or the Syracuse problem.

Take any natural number p. If p is even, divide it by 2 to get $\frac{p}{2}$ and if p is odd, multiply it by 3 and add 1 to obtain $3p+1$. Repeat the process indefinitely. Thus you get a sequence u starting with $u_0 = p$ and

$$\begin{cases} u_{n+1} &= \frac{u_n}{2} \text{ if } U_n \text{ is even ;} \\ u_{n+1} &= 3u_n + 1 \text{ if } U_n \text{ is odd ;} \end{cases}$$

for any natural number n.

For instance, starting with $p = 6$, one gets the sequence $u_0 = 6$, $u_1 = 3$, $u_2 = 10$, $u_3 = 5$, $u_4 = 16$, $u_5 = 8$,... The greatest value of that sequence is 16 (for u_4), and we have $u_8 = 1$. If we start with $p = 27$, then $u_0 = 27$, $u_1 = 82$, $u_2 = 41$, ... and for this sequence, the greatest value is $u_{77} = 9232$ and the first 1 is obtained with u_{111}.

The Collatz conjecture states that no matter what number you start with, you will always eventually reach 1. The smallest i such that $u_i = 1$ is called the *total stopping time* of the sequence.

Although the conjecture has not been proven, most mathematicians who have looked into the problem think the conjecture is true. The conjecture has been checked by computer for all starting values up to $20 \times 2^{58} \approx 5.764 \times 10^{18}$. Such computer evidence is not a proof that the conjecture is true. Since sequentially examining all natural numbers is a process which can never be completed, such an approach can never demonstrate that the conjecture is true, merely that no counterexamples have yet been discovered.

Adapted from Wikipedia, the free encyclopedia.

Questions

1. Can we assert that any sequence built as in the text will reach the value of 1? Explain your answer.
2. Compute the twelve first values of the sequence starting with $p = 6$ and check that its total stopping time is 8.
3. Explain why, after a value of 1, the sequence will repeat with the cycle $4 - 2 - 1$.
4. Find the greatest value and stopping time when the sequence starts from $p = 35$.
5. Can you find sequences where the greatest term is equal to the first term ?