Épreuve de section européenne

The cosine rule

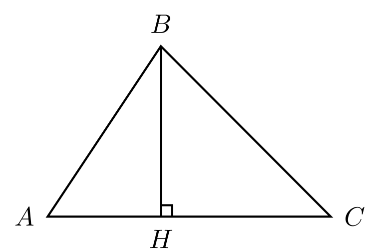
The cosine rule generalizes the Pythagorean theorem which holds only for right-angled triangles. Let ABC be any triangle where a, b and c are the lengths of the sides opposite the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$. This rule states that:

$$c^2 = a^2 + b^2 - 2ab\cos(\angle ACB). \tag{1}$$

Though the notion of cosine was not yet developed in his time, Euclid's *Elements*, dating back to the 3rd century BC, contains an early geometric theorem equivalent to the cosine rule. The case of obtuse triangle and acute triangle (corresponding to the two cases of negative or positive cosine) are treated separately, in Propositions 12 and 13 of Book 2.

Proposition 13: In acute-angled triangles, the square on the side opposite the acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by one of the sides about the acute angle, namely that on which the perpendicular falls, and the straight line cut off within by the perpendicular towards the acute angle.

This Proposition is illustrated on the right with an acute angle at C, and the line BH perpendicular to the line AC. It means that the square on AB is less than the sum of the squares on CA and CB by twice the rectangle CA by CH.



Adapted from Wikipedia and from D.E. Joyce's website at Clark University, Massachusetts.

Questions

1. Suppose that the triangle ABC is right-angled at C. Write the Pythagorean theorem for that triangle.
2. Why can we consider the cosine rule as a generalization of the Pythagorean theorem?
3. Prove that if $\angle ACB$ is an acute angle in the triangle ABC, Euclid's Proposition 13 implies the cosine rule.
4. Let ABC be a triangle with $\angle ACB = 60°$, $BC = 16$ and $AC = 21$. Compute the value of the length AB.
5. Prove the cosine rule (1) for any triangle, even with an obtuse angle at $\angle ACB$ (you may use the scalar product and expand $(\overrightarrow{AC} + \overrightarrow{CB})^2$).