Épreuve de section européenne

Erdös-Straus conjecture

A unit fraction $\frac{1}{n}$ is a fraction with numerator 1 and denominator any natural number n (with $n \geq 1$). The Egyptians used to work with such fractions. The Rhind papyrus dating to around 1650 BC gives a table of fractions $\frac{2}{n}$, where n is a natural number, written as the sum of unit fractions. For example, we get $\frac{2}{15} = \frac{1}{10} + \frac{1}{30}$ or $\frac{2}{19} = \frac{1}{12} + \frac{1}{76} + \frac{1}{114}$.

In the same way, $\frac{4}{n}$ can be expressed as the sum of four unit fractions since $\frac{4}{n} = \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \frac{1}{n}$. But is it possible to write $\frac{4}{n}$ as a sum of only three unit fractions, where the denominators can be equal?

In 1948, two mathematicians, Erdös and Straus, formulated an hypothesis (or conjecture) which states that for every natural number n with $n \geq 4$, the fraction $\frac{4}{n}$ can be written as the sum of three unit fractions. This means that for any integer $n \geq 4$, there are three natural numbers x, y and z such that $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$.

For instance, we have: $\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$ and we say that $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$ is an expansion of $\frac{4}{5}$.

This conjecture has been proved for some categories of numbers n, but not for every number n. For example, if n is not prime with $n = pq$ where p and q are whole numbers (different from 1 and n) and if we know an expansion of $\frac{4}{p}$ as a sum of three unit fractions, we can deduce an expansion of $\frac{4}{pq}$ as a sum of three unit fractions. For instance, we get: $\frac{4}{15} = \frac{1}{6} + \frac{1}{15} + \frac{1}{30}$ due to the expansion of $\frac{4}{5}$.

Computer searches have verified the truth of the conjecture up to $n = 10^{14}$, but proving it for all n remains an open problem.

Adapted from Wikipedia and other sources.

Questions

1. Check the decomposition of $\frac{2}{19}$ given by the Rhind papyrus.
2. There exists a simple algorithm to expand a fraction $\frac{p}{q}$ less than 1 into unit fractions: Find the greatest unit fraction less than $\frac{p}{q}$, say $\frac{1}{m}$. Find the fraction $\frac{p'}{q'}$ such that, $\frac{p}{q} = \frac{1}{m} + \frac{p'}{q'}$ and apply the algorithm to $\frac{p'}{q'}$. Go on until the process ends.
   (a) Use this algorithm to find an expansion of $\frac{4}{9}$ as a sum of unit fractions, and deduce a decomposition in three unit fractions.
   (b) Use this algorithm to find a second expansion of $\frac{4}{5}$, different from $\frac{1}{2} + \frac{1}{5} + \frac{1}{10}$.
3. Explain why if we know an expansion of $\frac{4}{p}$ as a sum of three unit fractions, we can write $\frac{4}{pq}$ as a sum of three unit fractions (with p and q whole numbers, $p \geq 4$).
4. Explain why, when looking for a counterexample to the Erdös-Straus conjecture, you just have to study prime denominators.