Épreuve de section européenne

Odd and even functions

An *even* function of x is one which is unaltered when x changes sign ; e.g. $\cos x$, $x^4 + x^2 + 2$, and any algebraical expression which contains only even powers of x.

Generally, if $f(x)$ is an even function of x, then $f(-x) = f(x)$.

An *odd* function of x is one which merely changes sign when x changes sign ; $\sin x$, $x^3 + 3x$, and any algebraical expression which consists entirely of odd powers of x.

Generally, if $f(x)$ is an odd function of x, then $f(-x) = -f(x)$.

An odd function of x must vanish when $x = 0$; for if

$$f(-x) = -f(x)$$

then, when $x = 0$, we have $f(0) = -f(0)$, therefore $f(0) = 0$.

Functions such as $3 \sin x + 2 \cos x$, $x^3 + 3x + 10$, in which some of the terms change sign and some are unchanged when x is replaced by $-x$, are neither even nor odd.

Adapted from *Introduction to Infinitesimal Calculus*, G.W. Caunt, 1931

Questions

1. Check that the function $x^4 + x^2 + 2$ is even and that the function $x^3 + 3x$ is odd.
2. Draw a trigonometric circle to show that the cosine function is even and that the sine function is odd.
3. (a) Explain the sentence “An odd function of x must vanish when $x = 0$”. Check that it is true for the odd functions mentioned in the text.
   (b) If a function f is such that $f(0) = 0$, is it necessarily an odd function ? If not, give a counter-example.
4. Prove that any polynomial function of x that contains only even powers of x is even.
5. Check that the functions $3 \sin x + 2 \cos x$, $x^3 + 3x + 10$ are neither even nor odd.
6. The graph of any even function exhibits a special geometrical property. Find out what it is, and prove it. Do the same for the graph of any odd function.