# Épreuve de section européenne

**The Shoelace Problem**

There are at least three common ways to lace shoes: American zigzag, European straight and quick-action shoe store.


To the purchaser, styles of lacing can differ in the time required to tie them, but to the shoe manufacturer, a more pertinent concern is which type of lacing requires the shortest -and therefore cheapest-laces; in the following paragraphs, we will ask the shoe manufacturer's question. We focus only on the length represented by straight line segments. The amount of extra lace required to tie an effective bow is the same for all methods of lacing, so it can be ignored.

One can calculate the length of the lace in terms of three parameters: the number $n$ of pairs of eyelets, the distance $d$ between successive eyelets and, the gap $g$ between corresponding left and right eyelets. It is not too hard to show that the lengths for the lacings are as follows:

**American:** $g + 2(n-1)\sqrt{d^2 + g^2}$

**European:** $(n-1)g + 2\sqrt{d^2 + g^2} + (n-2)\sqrt{4d^2 + g^2}$

**Shoe store:** $(n-1)g + (n-1)\sqrt{d^2 + g^2} + \sqrt{(n-1)^2 d^2 + g^2}$

Some careful high school algebra shows that if $d$ and $g$ are nonzero and $n$ is at least 4, then the shortest lacing is always American, followed by European, followed by shoe store. If $n = 3$, American is still shortest, but European and shoe-store lacings are of equal length. If $n = 2$, then all three lacings are equally long, but only a mathematician would worry about such cases!

Adapted from Ian Stewart, *Arithmetic and old laces*, Scientific American, July 1996

## Questions

1. Which way do you lace your shoes? Did you know the three different methods mentioned in the text?
2. (a) Explain the three formulae.
   (b) Supposing that $d$ and $g$ are fixed, are the three formulae linear functions of the number $n$ of eyelets?
3. Case $n = 2$: prove that all three formulae give the same answer.
4. Case $n = 3$: prove that the "American length" is less than the other ones, which are equal.
5. Case $n \geq 4$: prove that the "American length" is less than the "European length" (you can consider $d$ and $g$ constant and express both formulae as functions of $n$, then prove that the difference between them is an increasing function of $n$).