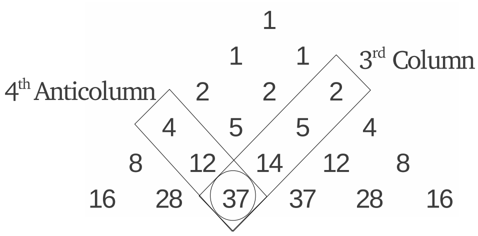
Épreuve de section européenne

A variant of Pascal's triangle

This variant has the rule that every entry, denoted as $a_{n,k}$ where $a_{1,1} = 1$, is calculated in a way such that, when these numbers are put into a triangular formation, every number is the sum of all the numbers above it in its two diagonals. The beginning of the triangle looks like the following in which for instance, the leftmost 37 is obtained as the sum $4 + 12 + 2 + 5 + 14$.



The *rows* of the triangle will begin at $n = 1$ and work down the triangle in increments of 1.

The *columns* of the triangle point southwest (60° from horizontal) and will begin at $k = 1$, and proceed in increments of 1. For example, we would say that the leftmost number 37 would be in row 6 and in column 3.

The *anticolumns* are the columns of the triangle pointing southeast that begin at $k' = 1$, and proceed in increments of 1 moving diagonally from right to left. So, for example, the leftmost number 37 would be in anticolumn 4.

We know an interesting property about the sum of the elements in row n of an ordinary Pascal's triangle. Is there any such property here? In fact, if the sum of the elements in any row n is represented by S_n, we have: $S_1 = 1$ and $S_n = \frac{2}{9}3^n$ for $n \geq 2$. Let's prove it.

Lemma 1: $S_n = 2S_{n-1} + 2S_{n-2} + \ldots + 2S_1$, for $n \geq 2$.

Proof: Choose an arbitrary element $a_{r,k}$. Consider the n-th row, where $n > r$. Then, $a_{r,k}$ will appear in the formula for finding $a_{n,k}$ because it is in the same column. Further, $a_{r,k}$ will appear in the formula for finding $a_{n,k+n-r}$ because of the fact that $a_{r,k}$ is in its anticolumn. Hence, the arbitrary element $a_{r,k}$ is summed exactly twice in the row n for $1 \leq r \leq n - 1$. And so, the sum of the elements in row n would be equal to twice the sum of the elements in all previous rows.

Lemma 2: $S_n = 3S_{n-1}$, for $n \geq 3$.

Proof: It can be shown from Lemma 1 that $Sn = 2S_{n-1} + S_{n-1}$. This means $S_n = 3S_{n-1}$.

The formula is a direct consequence of Lemma 2.

Adapted from Dennis Van Hise's research proposal, Stetson University, November 2000

Questions

1. Compute the numbers in row 7 and in row 8.
2. What is the number of the anticolumn which contains $a_{n,k}$?
3. Check the S_n formula, for $n = 3$, 4, and 7.
4. Focus at the proof of lemma 1:
   (a) Explain why $a_{4,2}$ appears twice in the sum of row 6.
   (b) Why is $a_{r,k}$ appearing in the formula for finding $a_{n,k+n-r}$ (with $r < n$)?
5. Proof of lemma 2: "It can be shown..." is rather elusive. Explain it.
6. Complete the proof of the theorem.