Épreuve de section européenne

Farey sequence

In mathematics, the Farey sequence of order n is the sequence of completely reduced fractions between 0 and 1 which, when in lowest terms, have denominators less than or equal to n, arranged in order of increasing size.

Each Farey sequence starts with the value 0, denoted by the fraction $\frac{0}{1}$, and ends with the value 1, denoted by the fraction $\frac{1}{1}$ (although some authors omit these terms).

For example, the Farey sequence of order 4 is denoted F_4 and is

$$F_4 = \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$

There are many interesting things about Farey sequences but we will focus on how to build F_n, the Farey sequence of order n, when n is a given natural number. Here is a way to do it :

The Farey sequence of order n contains all of the terms of the Farey sequences of lower orders. In particular F_n contains all of the terms of F_{n-1}. So knowing F_{n-1}, to get F_n, you only have to add the irreducible fractions between 0 and 1 whose denominator is n.
For example, F_6 consists of F_5 together with the fractions $\frac{1}{6}$ and $\frac{5}{6}$.

From various sources

Questions

1. Using the definition, give the Farey sequence of order 3.
2. Using the method explained above, deduce F_5 from F_4.
3. Explain why the Farey sequence of order n contains all of the terms of the Farey sequence of order $n-1$.
4. Another method to get F_n from F_{n-1} is to use the following property : If $\frac{a}{b}$, $\frac{p}{q}$ and $\frac{c}{d}$ are three consecutive terms in a Farey sequence such that $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ then $\frac{p}{q} = \frac{a+c}{b+d}$. Illustrate this property with the values from F_4,
5. Admitting that $\frac{a}{b}$ and $\frac{c}{d}$ are two consecutive terms in a Farey sequence if and only if $bc - ad = 1$, prove the property given in the previous question.