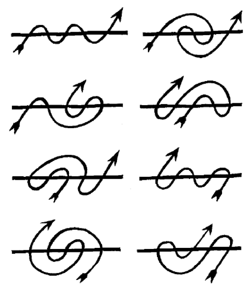
Épreuve de section européenne

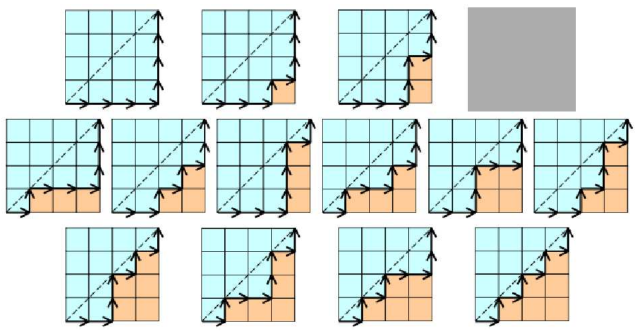
Meandric and Catalan Numbers

Given a fixed oriented line (a road), the meandric problem asks: in how many different ways can a river (starting in the South-West and flowing North-East) cross a road n times? For $n = 5$ crossings there are $M_5 = 8$ possibilities, shown in the figure on the right.



The sequence M_1, M_2, M_3,... begins: 1, 1, 2, 3, 8, 14, 42, 81, 262, 538... These are called Meandric numbers, since the river meanders across the road.

Nobody has found a formula which gives the n-th Meandric number. Mathematicians just achieved to find a sequence which gives an upper bound, called the Catalan sequence, named after the Belgian mathematician Eugène Catalan (1814-1894).



There are many different definitions of the Catalan numbers. For instance, C_n can be defined as the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. The figure on the left gives 13 of the 14 possibilities (one is hidden).

The formula for Catalan numbers C_n is difficult to prove but well-known : for $n \geq 1$, $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$.

Adapted from various sources

Questions

1. Explain why the first Meandric number is equal to 1.
2. What link between Meandric numbers and Catalan numbers is mentioned in the text?
3. Draw two different pictures to illustrate the Meandric number M_3, which is equal to 2.
4. (a) Find the 14th path, which is hidden in the picture.
   (b) The first Catalan numbers are 1, 2, 5, 14, 42, 132, 429. Use the formula given in the text to compute the next one.
   (c) Check that for $n \leq 8$, $M_n \leq C_n$.
   (d) Prove that $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$.