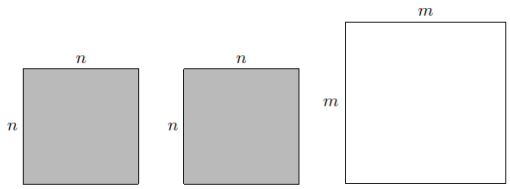
Épreuve de section européenne

Conway Proof

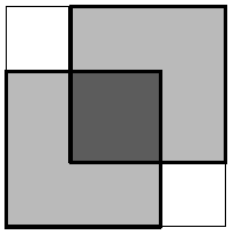
I'd like to take you through some simple ideas relating to squares. Let's start with a new proof of an old theorem. "Is the square root of two a ratio of two whole numbers?" Let's put the question another way. Could there be two squares with side equal to a whole number n whose total area is identical to that of a single square with side equal to another whole number m?



This nearly happens for the sum of two 12 by 12 squares, which gives approximately a 17 by 17 square. So $17/12 = 1.41666\ldots$ is very close to the square root of two.

But we're not asking if you can find whole numbers m and n that roughly[1] satisfy $m^2 = 2n^2$. We want to establish if it can be done exactly!

Let's assume that it can be done. Then there must be a whole number m, which is the smallest for which it can be done. Let's draw a picture using that smallest possible m and let's stick the two small grey squares in the top right and bottom left corners of the big square. Now, part of the big square is covered twice, and part of the big square isn't covered at all, by the smaller squares. The part that's covered twice is shown in dark grey, and the bits that are not covered are shown in white. Since the area of the original big white square is exactly equal to the total area of the light grey squares, the area of the region that's covered twice must be exactly equal to the area of the bits that are not covered.



Finally, what are the sizes of these three areas? It's easy to prove that $2(m-n)^2 = (m-2n)^2$. We have just found another pair of squares whose sum of areas is the area of a new square. But this is impossible! Conclusion: the square root of two isn't rational.

Adapted from *The Power of Mathematics*, by John Conway

Questions

1. What is a whole number? What is a rational number?
2. (a) Check that $m = 17$ and $n = 12$ roughly satisfy $m^2 = 2n^2$ then explain the approximation of $\sqrt{2}$ given in the text.
   (b) Check that $m = 99$ and $n = 70$ also give a good approximation of $\sqrt{2}$.
3. Explain why $2(m-n)^2 = (m-2n)^2$.
4. Explain the sentence "But this is impossible!".
5. What are the whole numbers k such that $m^2 = kn^2$ has solutions? Give some examples and the solutions of the related equations.

[1] roughly : approximately