Épreuve de section européenne

The Lazy Caterer's Sequence

The lazy caterer's[1] sequence, more formally known as the central polygonal numbers, describes the maximum number of pieces of a circle (a pancake is usually used to describe the situation) that can be made with a given number of straight cuts. For example, three cuts across a pancake will produce six pieces if the cuts all meet at a common point, but seven if they do not. This problem can be formalized mathematically as one of counting the cells in an arrangement of lines. When a circle is cut n times to produce the maximum number of pieces, noted $f(n)$, the n-th cut must be considered; the number of pieces before the last cut is $f(n-1)$ while the number of pieces added by the last cut is n.

To obtain the maximum number of pieces, the cut line should cross all the other previous cut lines inside the circle, but not cross any intersection of previous cut lines. Thus, the n-th line itself is cut in $n-1$ places, and into n line segments. Each segment divides one piece of the $n-1$ cut pancake into 2 parts, adding exactly n to the number of pieces. Finally, the recurrence relation $f(n) = n + f(n-1)$ can be found.

Since $f(0) = 1$, it's easy to prove that $f(n) = n+(n-1)+(n-2)+\ldots+3+2+1+1$, which can also be written as $f(n) = 1+1+2+3+\ldots+n$. Finally, this is equal to the sum of a famous sequence: $f(n) = 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}$, which can also be expressed as $f(n) = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$.

From various sources

Questions

1. Explain why three cuts in a pancake usually produce six pieces instead of seven in the picture above.
2. Explain why $f(0) = 1$.
3. Why are there two "1" at the end of the formula $f(n) = n+(n-1)+(n-2)+\ldots+3+2+1+1$?
4. $\frac{n(n+1)}{2}$ is the sum of the n-th first consecutive terms of a famous sequence. Which one? Explain this calculation.
5. Here are the first twenty terms of the lazy caterer's sequence: 1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191. Explain how to find the 21st term without using the formula, then check it using the formula.
6. Prove that $f(n) = \binom{n}{2} + \binom{n}{1} + \binom{n}{0}$.

[1] caterer : someone hired to prepare food for a special occasion