Épreuve de section européenne

Gaussian Integers

In number theory, a Gaussian integer is a complex number whose real and imaginary parts are both integers. For instance, $3+2i$ is a Gaussian integer, but $0.2+3i$ is not.

Let x be a Gaussian integer. The four Gaussian integers x, ix, $-x$, and $-ix$ are called the *associates* of x. The figure corresponding to these four numbers is easy to visualize.

The addition of two Gaussian integers is a Gaussian integer, and it also works for multiplication. Of course, it doesn't work all the time with division: a Gaussian integer a is said to be divisible by another one b if there exists another Gaussian integer c such that $bc = a$. For instance, $a = 3-i$ is divisible by $b = 1-2i$ because the Gaussian integer $c = 1+i$ is such that $bc = (1-2i)(1+i) = 3-i = a$.

Let $\mathcal{C}$ be the circle with centre O and radius r. How many integer lattice[1] points are there inside $\mathcal{C}$? The first breakthroughs towards a solution were made by Carl Friedrich Gauss.

Adapted from various sources

Questions

1. Is $0.5+3i$ a Gaussian integer ? Explain why.
2. (a) Give the associates of $2+3i$ and use them to explain the sentence "The figure [...] is easy to visualize".
   (b) In what case is the figure a square ?
3. Prove that the multiplication of two Gaussian integers is a Gaussian integer.
4. Prove that $7+6i$ is divisible by $2+i$.
5. (a) What is the link between Gauss's circle problem and Gaussian integers ?
   (b) How many integer lattice points are there in the circle $\mathcal{C}$ centred in O and with radius 2?

[1] lattice = square grid