Épreuve de section européenne

Improper integrals

An integral which has an infinite (positive or negative) limit of integration is called an improper integral.

Example: $\int_0^{+\infty} f(t)\mathrm{d}t$ which reads "integral of f of t (ranging) from 1 to infinity".

Solving improper integrals requires a new technique: The problem with improper integrals arises from the fact that we cannot simply substitute infinity into a function and evaluate. So what we do is replace the indefinite limit(s) with a variable (say b), then evaluate the integral as usual and then take the limit of the result as b approaches infinity. For example:

$$\int_0^{+\infty} f(t)\mathrm{d}t = \lim_{b\to+\infty}\int_0^{b} f(t)\mathrm{d}t.$$

If the limit in any of the above integrals exists and is finite then we say that the integral converges, otherwise, if the limit is infinite or does not exist, we say that the integral diverges.

Example: To show that the following improper integral converges:

$$\int_1^{+\infty} e^{-3x}\mathrm{d}x$$

we use the process described above:

$$\int_1^{+\infty} e^{-3x}\mathrm{d}x = \lim_{b\to+\infty}\int_1^{b} e^{-3x}\mathrm{d}x = \lim_{b\to+\infty}\left[\frac{e^{-3x}}{-3}\right]_1^b = \lim_{b\to+\infty}\left[\frac{e^{-bx}}{-3}+\frac{1}{3}\right] = 0+\frac{1}{3} = \frac{1}{3}.$$

Adapted from *A course in Calculus, Unit 6*, University of New Brunswick

Questions

1. Give a method to evaluate an integral.
2. The text says: "if the limit is infinite or does not exist". Give an example of a function whose limit does not exist.
3. Show that the following improper integral diverges:

$$\int_1^{+\infty} \frac{1}{x}\mathrm{d}x.$$

4. Show that the following improper integral converges:

$$\int_1^{+\infty} \frac{1}{x^2}\mathrm{d}x.$$