Épreuve de section européenne

Proof by induction

"If there are n coins in a purse, and if at least one of them is a gold one, then all of them are gold coins."

Let us prove this curious assertion by induction :

Basis : if $n = 1$, the property is obvious.

Inductive step : Let us assume that the property is true for some given positive integer n.

Consider a purse with $n + 1$ coins in it, and with at least a gold coin. Let X be that coin. If we remove a coin Y, other than X, we obtain a purse that satisfies the induction hypothesis (n coins including a gold one). Thus, the purse contains n gold coins.

We will now prove that Y is also a gold coin : let us replace one of the gold coins, other than X, by Y. Again, we obtain a purse that satisfies the induction hypothesis and we can deduce that all the coins, including Y, must be gold coins. Therefore, all the $n + 1$ coins are gold ones. The property is true with $n + 1$ coins.

Conclusion : Since both the basis and the inductive step have been proved, the property is true for any positive integer n.

Adapted from various sources

Questions

1. (a) What do you think of the assertion ?
   (b) What do you think of the proof ?
   (c) Focus on the case $n = 2$. Is the inductive step valid for that value of n ?
2. Prove by induction that for any positive integer n, $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.
3. What other types of proof do you know ?