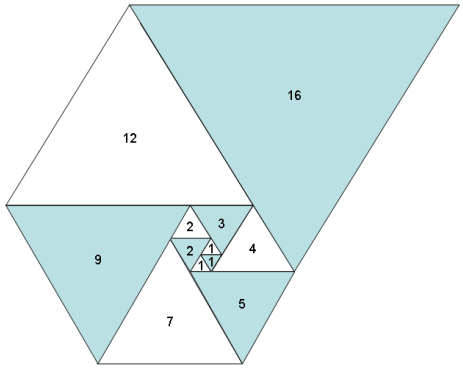
Épreuve de section européenne

The plastic number

The plastic number is a little-known relative of the famous golden number. Just as the Fibonacci numbers create a spiraling system of squares, related to the golden number, there is a similar spiral diagram for the plastic number, but composed of equilateral triangles. In the diagram below, the initial triangle has side 1 and successive triangles spiral in a clockwise direction: the spiral shown is again roughly logarithmic. In order to make the shapes fit; the first three triangles all have side 1. The next two have side 2, and then the numbers go 4, 5, 7, 9, 12, 16, 21, and so on.



This sequence is known as the Padovan sequence. Again there is a simple rule for finding these numbers, analogous to that for Fibonacci numbers: each number in the sequence is the sum of the previous number but one, together with the one before that. For example, $12 = 7 + 5$, $16 = 9 + 7$, $21 = 12 + 9$.

This pattern follows from the way the triangles fit together. If P_n is the n-th Padovan number (starting from $P_1 = P_2 = P_3 = 1$), then $P_n = P_{n-2} + P_{n-3}$.

The plastic number, which we denote by p, is the limit of the ratios of successive Padovan numbers. That is, ratios of successive Padovan numbers, such as 49/37 or 151/114, give good approximations of the plastic number.

The pattern of the sequence of Padovan numbers leads to the equation $p^3 - p - 1 = 0$, and p is the unique real solution of this cubic equation.

Adapted from *Cabinet of Mathematical Curiosities* by Ian Stewart

Questions

1. Find out the values of P_2 and P_{11}.
2. Compute P_{14} and P_{15}.
3. What is the link between the Padovan sequence and the plastic number?
4. Give the 2 approximate values of p mentioned in the text.
5. Use a calculator to find an approximate value of p to 3 decimal places. Check that it is almost a solution of the cubic equation $p^3 - p - 1 = 0$.