Épreuve de section européenne

Perfect and amicable numbers

When a number P is equal to the sum of its proper divisors (the divisors of a number, not including the number itself), this number is called a perfect number. For instance, 6 is a perfect number since the proper divisors of 6 are 1, 2, and 3, and $6 = 1 + 2 + 3$.

In a romantic short story, *Mathematical Aphrodisiac* by Alex Galt, a related concept appears:

> Did you ever hear of amicable numbers? They're like perfect numbers, but instead of being the sum of their own divisors, they're the sum of each other's divisors. In the Middle Ages people used to carve amicable numbers into pieces of fruit. They'd eat the first piece themselves and then feed the other one to their lover. It was a mathematical aphrodisiac. I love that – a mathematical aphrodisiac.

The smallest amicable (or friendly) pair has been known since Antiquity. It was not until 1636 that the great Fermat discovered another pair of friendly numbers: 17296 and 18416. In 1866, a smaller pair, 1184 and 1210 was announced by Nicolo Paganini, a 16-year old Italian. This pair had even been overlooked by Euler who drew up a list of 64 friendly pairs in the 18th century from which two turned out to be unfriendly. Today about 12 million pairs of friendly numbers are known.

Adapted from Simon Singh's website and other sources

Questions

1. Find a perfect number between 20 and 30.
2. Prove that 496 is a perfect number.
3. Is there a pair of friendly numbers between 20 and 30?
4. Check that 220 and 284 are friendly.
5. We admit that if n is an integer such that the numbers $a = 3 \times 2^n - 1$, $b = 3 \times 2^{n-1} - 1$ and $c = 3^2 \times 2^{n-1} - 1$ are all prime, then $2^n ab$ and $2^n c$ are friendly.
   Find out the values of the friendly numbers given by these formulas when $n = 4$.